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FINITE DOMINATION AND NOVIKOV RINGS. LAURENT POLYNOMIAL RINGS IN SEVERAL VARIABLES

THOMAS HÜTTEMANN AND DAVID QUINN

ABSTRACT. We present a homological characterisation of those chain complexes of modules over a LAURENT polynomial ring in several indeterminates which are finitely dominated over the ground ring (that is, are a retract up to homotopy of a bounded complex of finitely generated free modules). The main tools, which we develop in the paper, are a non-standard totalisation construction for multi-complexes based on truncated products, and a high-dimensional mapping torus construction employing a theory of cubical diagrams that commute up to specified coherent homotopies.

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Introduction

Let $R \subseteq K$ be a pair of unital rings. A cochain complex C of K -modules is called *R -finitely dominated* if C is homotopy equivalent, as an R -module complex, to a bounded complex of finitely generated projective R -modules; equivalently, if C is a retract up to homotopy of a bounded complex of finitely generated free R -modules [Ran85, Proposition 3.2].

Finite domination is relevant, for example, in group theory and topology. Suppose that G is a group of type (FP) ; this means, by definition, that the trivial G -module \mathbb{Z} admits a finite resolution C by finitely generated projective $\mathbb{Z}[G]$ -modules. Let H be a subgroup of G . Deciding whether H is of type (FP) is equivalent to deciding whether C is $\mathbb{Z}[H]$ -finitely dominated.

In topology, finite domination has been considered in the context of homological finiteness properties of covering spaces (DWYER and FRIED [DF87]), or properties of ends of manifolds (RANICKI [Ran95]).

Our starting point is the following result of RANICKI [Ran95, Theorem 2]: *Let C be a bounded complex of finitely generated free modules over $K = R[x, x^{-1}]$. The complex C is R -finitely dominated if and only if the two complexes*

$$C \otimes_K R((x)) \quad \text{and} \quad C \otimes_K R((x^{-1}))$$

are acyclic. Here $R((x)) = R[[x]][x^{-1}]$ denotes the ring of formal LAURENT series in x , and $R((x^{-1})) = R[[x^{-1}]]x$ denotes the ring of formal LAURENT series in x^{-1} .

For LAURENT polynomial rings in several indeterminates, it is possible to strengthen this result to allow for iterative application, see for example [HQ13]. In particular, writing $L = R[x, x^{-1}, y, y^{-1}]$ for the LAURENT polynomial ring in two variables, one can show that a bounded complex of finitely generated free L -modules is R -finitely dominated if and only if the four complexes

$$\begin{array}{ll} C \otimes_L R[x, x^{-1}]((y)) & C \otimes_L R[x, x^{-1}]((y^{-1})) \\ C \otimes_{R[x, x^{-1}]} R((x)) & C \otimes_{R[x, x^{-1}]} R((x^{-1})) \end{array}$$

are acyclic.

This characterisation has the rather unsatisfactory feature that the tensor products are taken over two different rings. In the present paper, we propose a different, non-iterative approach leading to an entirely new characterisation of finite domination. Roughly speaking, a cone in \mathbb{R}^n determines a certain ring of formal LAURENT series, and our main theorem asserts that finite domination is equivalent to the vanishing of homology with coefficients in these LAURENT series rings for all cones σ in a complete fan.

INFORMAL STATEMENT OF RESULTS

We think of the LAURENT polynomial ring $L = R[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$ in n indeterminates as the monoid ring $R[\mathbb{Z}^n \cap \mathbb{R}^n]$, identifying the n -tuples of exponents of monomials with integral points in \mathbb{R}^n . A cone $\sigma \subseteq \mathbb{R}^n$ then defines a subring $R[\mathbb{Z}^n \cap \sigma^\vee]$ of L , where σ^\vee is the dual cone. Another way to describe this is as follows. The support of a formal sum $f = \sum_{\mathbf{a} \in \mathbb{Z}^n} r_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$, where $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$, is the set of those $\mathbf{a} \in \mathbb{Z}^n$ with $r_{\mathbf{a}} \neq 0$. Then $R[\mathbb{Z}^n \cap \sigma^\vee]$ is the set of all such f having finite support which is contained in σ^\vee . Omitting the finiteness condition yields a set of formal LAURENT series; if $\dim \sigma = n$ the ring $R((\sigma))$ can be thought of as the set of those f with support in a translated copy of σ^\vee , where we allow translation by the negative of a vector in the interior of σ^\vee . This construction is modified for cones of dimension less than n ; the modification, and the ring structure of $R((\sigma^\vee))$, are explained in detail in §III.6.

For example, if σ is a cone spanned by d elements of \mathbb{Z}^n which are \mathbb{Z} -linearly independent, then there is an R -algebra isomorphism

$$R((\sigma)) \cong R[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_{n-d}^{\pm 1}][[x_{n-d+1}, x_{n-d+2}, \dots, x_n]] [1 / \prod_{k=0}^{d-1} x_{n-k}]$$

between $R((\sigma))$ and a localisation of a formal power series ring over a LAURENT polynomial ring; the right-hand side consists of formal LAURENT series (with coefficients in a ring of LAURENT polynomials) having the property that the support lies in the “first orthant” after shifting along the diagonal vector $(1, 1, \dots, 1)$ finitely often.

In Theorems III.5.1 and III.6.4 we show that if C is a bounded complex of finitely generated free L -modules which is R -finitely dominated then the induced complex $C \otimes_L R((\sigma^\vee))$ is acyclic. We also show that, conversely, if $C \otimes_L R((\sigma^\vee))$ is acyclic for all σ coming from a finite family of cones covering all of \mathbb{R}^n , then C is necessarily R -finitely dominated.

For two variables ($n = 2$) a version of this programme has been carried out by the authors in the paper [HQ14]. The present extension to more than two variables is non-trivial as it demands a theory of high-dimensional mapping tori, in turn resting on a theory of homotopy commutative cubical diagrams. Both are developed in this paper, and might be of independent interest for researchers in homological algebra.

STRUCTURE OF THE PAPER

The paper is divided into three parts. In the first we develop the theory of homotopy commutative cubical diagrams of cochain complexes, culminating in the construction of derived cubes (Theorem I.5.1) which are,

roughly speaking, homotopy commutative diagrams obtained from commutative ones by replacing all entries with homotopy equivalent ones. In the second part, we introduce higher-dimensional mapping tori, and prove an algebraic high-dimensional analogue of MATHER's trick of turning a mapping torus of a composite map through an angle of π (Lemmas II.2.6 and II.4.3). In the third part, we define a non-standard totalisation construction for multi-complexes based on truncated products rather than direct sums or products, and prove a simple vanishing criterion (Proposition III.2.4). The main result is then proved by a combination of multi-complex and mapping torus techniques.

CONVENTIONS

We fix some notation to be used throughout the paper. Let N be a totally ordered indexing set with n elements; we will mostly work with the set $N = \{1, 2, \dots, n\}$. The letters A , B and S will denote subsets of N with cardinalities a , b and s , respectively, unless explicitly defined otherwise. Let R be an arbitrary unital ring. Modules will always be right modules if not specified otherwise. Our complexes will be indexed cohomologically (differentials increase the degree), and will thus be termed cochain complexes. A cochain complex C is bounded above (*resp.*, bounded) if $C^n = 0$ for all $n \gg 0$ (*resp.*, $|n| \gg 0$). The cohomology modules of an arbitrary cochain complex C with differential d are defined as usual as the quotient modules

$$H^n(C) = (\ker(d: C^n \longrightarrow C^{n+1})) / (\operatorname{Im}(d: C^{n-1} \longrightarrow C^n)) .$$

A map of cochain complexes is called a quasi-isomorphism if it induces isomorphisms on all cohomology modules. A standard result in homological algebra asserts that a quasi-isomorphism between bounded-above cochain complexes of projective modules is a homotopy equivalence. We will use this result frequently throughout the paper.

Part I. Homotopy commutative cubes

A commutative square diagram of cochain complexes and cochain maps,

$$\begin{array}{ccc} C_2 & \xrightarrow{f_{12,2}} & C_{12} \\ f_{2,\emptyset} \uparrow & & \uparrow f_{12,1} \\ C_\emptyset & \xrightarrow{f_{1,\emptyset}} & C_1 \end{array}$$

can be considered as a three-fold cochain complex with commuting differentials; we think of the cochain direction as the last-coordinate direction, or z -direction, with the square sitting at coordinates $x = 0, 1$ and $y = 0, 1$. The totalisation T then is a cochain complex given by

$$T^n = C_\emptyset^n \oplus (C_1^{n-1} \oplus C_2^{n-1}) \oplus C_{12}^{n-2} ,$$

with differential described by the following matrix (suppressing zero entries):

$$D = \begin{pmatrix} d_\emptyset & & & \\ f_{1,\emptyset} & -d_1 & & \\ f_{2,\emptyset} & & -d_2 & \\ & -f_{12,1} & f_{12,2} & d_{12} \end{pmatrix}$$

The construction can be extended to higher-dimensional cubes in a standard manner. It is a different matter altogether what happens for diagrams that commute up to homotopy only. For the square above, suppose that H is a homotopy between $f_{12,1} \circ f_{1,\emptyset}$ and $f_{12,2} \circ f_{2,\emptyset}$. The above totalisation construction, if applied verbatim, fails to result in a cochain complex. However, if the matrix D is modified to include the specified homotopy

$$\begin{pmatrix} d_\emptyset & & & \\ f_{1,\emptyset} & -d_1 & & \\ f_{2,\emptyset} & & -d_2 & \\ \pm H & -f_{12,1} & f_{12,2} & d_{12} \end{pmatrix}$$

(the sign depending on the direction of the homotopy), then we obtain a cochain complex again, and it seems justified to consider this as the “right” totalisation construction for homotopy commutative squares equipped with a choice of homotopy.

In this first part of the paper, we *define* a homotopy commutative cube to be a collection of cochain maps, homotopies and higher homotopies with the characteristic property that a totalisation construction, similar to the one above and to be detailed below, results in a cochain complex. In fact, for reasons of aesthetics we go one step further: the data we consider consists of a collection of graded modules, together with module maps $H_{B,A}$, indexed by pairs $A \subseteq B$ of subsets of a given indexing set N , with $H_{B,A}$ being of degree $1 - \#A - \#B$. The main advantage is that the differentials (which correspond to the case $A = B$) are now treated in exactly the same way as all the other structure maps, leading to a slightly more symmetric definition of homotopy commutative cubes. It is then an easy exercise, solved in §I.3 below, to show that the data defining a homotopy commutative cube consists of cochain complexes, cochain maps and, for each two-dimensional face of the cube, homotopies between the cochain maps. Data associated to higher-dimensional faces should then be interpreted as higher homotopies, or coherence data.

We then specialise to homotopy commutative cubes which have the same cochain complex attached to each vertex, and in which the structure maps depend on the direction in the cube only (and not on their position within the cube). Here the main point is to realise that the definition of homotopy commutativity leads to a consistent, meaningful notion. This is recorded as Lemma I.4.2.

Finally, we include a non-trivial example of a homotopy commutative cube which is constructed from a commutative cubical diagram. Details are contained in Theorem I.5.1 and its proof. This construction of “derived” homotopy-commutative cubical diagrams will be an essential ingredient for the analysis of higher-dimensional mapping tori in later parts of the paper.

I.1. TOTAL INCIDENCE NUMBERS

Definition I.1.1. Let $B = \{b_1 < b_2 < \dots < b_b\} \subseteq N$ and write $d_i(B) = B \setminus \{b_i\}$, for $1 \leq i \leq b$. Given a subset $A \subset B$ there is a unique way to write $A = d_{i_1} d_{i_2} \dots d_{i_{b-a}}(B)$ with $i_1 < i_2 < \dots < i_{b-a}$, and we define

$$[B : A] = (-1)^{b-a} (-1)^{i_1 + i_2 + \dots + i_{b-a}}.$$

For $A \not\subseteq B$ we define $[B : A] = 0$, and we set $[A : A] = 1$ for any A . In either case we call $[B : A]$ the *total incidence number* of A and B .

From the simplicial identity $d_i \circ d_j(B) = d_{j-1} \circ d_i(B)$ for $i < j$ we immediately infer that

$$\begin{aligned} [A \amalg \{x, y\} : A \amalg \{x\}] \cdot [A \amalg \{x\} : A] \\ = -[A \amalg \{x, y\} : A \amalg \{y\}] \cdot [A \amalg \{y\} : A] \end{aligned} \quad (\text{I.1.2})$$

for distinct elements $x, y \in N \setminus A$. We also have the following combinatorial re-statement of the definition of total incidence numbers:

Lemma I.1.3. For $B \supseteq A$ we have $[B : A] = (-1)^\kappa$ where κ is the number of pairs $(b, x) \in B \times (B \setminus A)$ with $b < x$. In particular,

$$[B : \emptyset] = (-1)^{b(b-1)/2} = \begin{cases} 1 & \text{if } b \equiv 0, 1 \pmod{4} \\ -1 & \text{if } b \equiv 2, 3 \pmod{4} \end{cases}$$

and $[B : B \setminus \{z\}] = (-1)^{\#\{b \in B \mid b < z\}}$ for $z \in B$. \square

Lemma I.1.4. Given sets $B \supseteq S \supseteq A$ and an element $z \in N \setminus B$, the products of total incidence numbers¹

$$[B : S][S : A] \quad \text{and} \quad [B \amalg z : S \amalg z][S \amalg z : A \amalg z]$$

differ by a factor $\epsilon = \pm 1$ that is independent of S (that is, depends only on B , A and z). We also have $[B \amalg z : A \amalg z] = \epsilon \cdot [B : A]$, with the same factor ϵ .

Proof. Write $S = d_{i_1} d_{i_2} \dots d_{i_{b-s}}(B)$ with $i_1 < i_2 < \dots < i_{b-s}$, and $B = S \amalg \{z_1 < z_2 < \dots < z_{b-s}\}$. Then clearly $S \amalg z = d_{j_1} d_{j_2} \dots d_{j_{b-s}}(B \amalg z)$ where

$$j_\ell = \begin{cases} i_\ell & \text{if } z_\ell < z, \\ i_\ell + 1 & \text{if } z_\ell > z. \end{cases}$$

Consequently, $[B \amalg z : S \amalg z] = (-1)^\kappa [B : S]$ where κ is the number of elements in $B \setminus S$ which are bigger than z .

Now, re-defining some of the symbols above, write $A = d_{i_1} d_{i_2} \dots d_{i_{s-a}}(S)$ with $i_1 < i_2 < \dots < i_{s-a}$, and $S = A \amalg \{z_1 < z_2 < \dots < z_{s-a}\}$. Then clearly $A \amalg z = d_{j_1} d_{j_2} \dots d_{j_{s-a}}(S \amalg z)$ where

$$j_\ell = \begin{cases} i_\ell & \text{if } z_\ell < z, \\ i_\ell + 1 & \text{if } z_\ell > z. \end{cases}$$

Consequently, $[S \amalg z : A \amalg z] = (-1)^\nu [S : A]$ where ν is the number of elements in $S \setminus A$ which are bigger than z .

¹We simplify notation and write $A \amalg z$ instead of the more precise $A \amalg \{z\}$.

In total, the two products of incidence numbers differ thus by a factor $(-1)^{\kappa+\nu}$. But $\kappa + \nu$ is the number of elements in $(B \setminus S) \amalg (S \setminus A) = B \setminus A$ which are larger than z . This number does not depend on S .

The last assertion is the special case $S = B$ as $[B : A] = [B : B][B : A]$ and similarly $[B \amalg z : A \amalg z] = [B \amalg z : B \amalg z][B \amalg z : A \amalg z]$. \square

I.2. N -DIAGRAMS AND THEIR TOTALISATION

Definition I.2.1. An N -*diagram* consists of the following (non-functorial) data:

- for each $A \subseteq N$ a graded R -module $F(A) = \bigoplus_{k \in \mathbb{Z}} F(A)^k$;
- for each inclusion $A \subseteq B$ a graded module homomorphism

$$H_{B,A} : F(A) \longrightarrow F(B)$$

of degree $a - b + 1$.

We write $d = d_A$ in place of $H_{A,A}$, and for $b - a = 1$ we denote $H_{B,A}$ by $f_{B,A}$. (The data defining an N -diagram is not required to satisfy any compatibility conditions.)

Definition I.2.2. Let F be an N -diagram. The *totalisation* of F consists of the graded R -module $\text{Tot}(F)$ given by

$$\text{Tot}(F)^\ell = \bigoplus_{A \subseteq N} F(A)^{\ell-a} ,$$

and module homomorphisms

$$D(F) = D = (D_{B,A})_{A,B \subseteq N} : \text{Tot}(F)^\ell \longrightarrow \text{Tot}(F)^{\ell+1}$$

given by

$$D_{B,A} = \begin{cases} (-1)^{ab} [B : A] \cdot H_{B,A} & \text{if } A \subseteq B , \\ 0 & \text{otherwise.} \end{cases}$$

We can, and will, think of $D(F)$ as a matrix with columns and rows indexed by the subsets of N ; note that $D_{B,A} : F(A)^{\ell-a} \longrightarrow F(B)^{(\ell+1)-b}$ is a map of degree $a - b + 1$. The composition

$$D(F) \circ D(F) : \text{Tot}(F)^\ell \longrightarrow \text{Tot}(F)^{\ell+2}$$

is calculated by matrix multiplication; explicitly²,

$$(D(F) \circ D(F))_{B,A} = \sum_{A \subseteq \dot{S} \subseteq B} (-1)^{bs} (-1)^{sa} [B : S] [S : A] \cdot H_{B,S} \circ H_{S,A} . \quad (\text{I.2.3})$$

I.3. HOMOTOPY COMMUTATIVE N -CUBES

Definition I.3.1. A *homotopy commutative N -cube* is an N -diagram F such that its totalisation is a cochain complex of R -modules with differential $D(F)$ (that is, such that $D(F) \circ D(F) = 0$).

For the remainder of this section we consider a homotopy commutative N -cube F with totalisation $T = \text{Tot}(F)$ and associated differential $D = D(F)$.

²the variable over which the sum is taken is sometimes marked by a dot underneath

Lemma I.3.2. *The graded module $F(A)$ is a cochain complex of R -modules with differential d_A .*

Proof. By definition of a homotopy commutative N -cube we have $D \circ D = 0$, and so in particular, using (I.2.3), $d_A \circ d_A = (D \circ D)_{(A,A)} = 0$. \square

Lemma I.3.3. *For $b - a = 1$ the map $f_{B,A} = H_{B,A} : F(A) \longrightarrow F(B)$ is a cochain map.*

Proof. The (B, A) -entry of $D \circ D$ is given, according to (I.2.3), by

$$(-1)^{b^2}(-1)^{ba}[B : A] \cdot d_B \circ f_{B,A} + (-1)^{ba}(-1)^{a^2}[B : A] \cdot f_{B,A} \circ d_A .$$

Since $D \circ D = 0$, and since $(-1)^{b^2} = -(-1)^{a^2}$ this implies that $f_{B,A}$ is a cochain map as claimed. \square

Lemma I.3.4. *Let $B = A \amalg \{z_0 < z_1\}$, and write $Z_i = A \amalg \{z_i\} = B \setminus \{z_{1-i}\}$. Then $H_{B,A}$ is a homotopy from $f_{B,Z_0} \circ f_{Z_0,A}$ to $f_{B,Z_1} \circ f_{Z_1,A}$ so that*

$$d_B \circ H_{B,A} + H_{B,A} \circ d_A = f_{B,Z_1} \circ f_{Z_1,A} - f_{B,Z_0} \circ f_{Z_0,A} .$$

Proof. Again we will use that the (B, A) -entry of $D \circ D$ has to be trivial. That is, we must have

$$\begin{aligned} & (-1)^b(-1)^{ba}[B : A] \cdot d_B \circ H_{B,A} \\ & + (-1)^{b(b-1)}(-1)^{(a+1)a}[B : Z_0][Z_0 : A] \cdot f_{B,Z_0} \circ f_{Z_0,A} \\ & + (-1)^{b(b-1)}(-1)^{(a+1)a}[B : Z_1][Z_1 : A] \cdot f_{B,Z_1} \circ f_{Z_1,A} \\ & + (-1)^{ba}(-1)^a[B : A] \cdot H_{B,A} \circ d_A = 0 . \end{aligned}$$

(We have used $(-1)^{k^2} = (-1)^k$ and $[S : S] = 1$.) Now $(-1)^b(-1)^{ba} = (-1)^{ba}(-1)^a = 1$ as a and b have the same parity, and we similarly have $(-1)^{b(b-1)}(-1)^{(a+1)a} = 1$.

Write $A = d_i d_j(B)$ with $i < j$. Then $[B : Z_0] = (-1)^j$ and $[Z_0 : A] = (-1)^i$ so that $[B : Z_0][Z_0 : A] = (-1)^{j+i} = [B : A]$.

But we also have $A = d_{j-1} d_i(B)$; this implies $[B : Z_1] = (-1)^i$ and $[Z_1 : A] = (-1)^{j-1}$ so that $[B : Z_1][Z_1 : A] = (-1)^{j+i-1} = -[B : A]$.

Cancelling the common factor of $[B : A]$ and re-arranging the terms now gives

$$d_B \circ H_{B,A} + H_{B,A} \circ d_A = f_{B,Z_1} \circ f_{Z_1,A} - f_{B,Z_0} \circ f_{Z_0,A} ,$$

proving the Lemma. \square

In this vein one can work out conditions on the maps $H_{B,A}$ for $b - a \geq 3$; these maps provide what are sometimes called higher (coherent) homotopies.

I.4. SPECIAL N -DIAGRAMS AND SPECIAL N -CUBES

Definition I.4.1. An N -diagram is called *special* if the graded R -module $F(A)$ does not depend on A (that is, if $F(A) = F(B)$ for all $A, B \subseteq N$), and if the maps $H_{B,A}$ depend only on $B \setminus A$. We will usually use the following notation for a special N -diagram:

- C is the graded R -module $F(A)$, for any $A \subseteq N$;
- $d = d_A$, for any $A \subseteq N$;

- $f_k = f_{\{k\}, \emptyset}$, for $k \in N$;
- $H_S = H_{B,A}$, for any $A \subseteq B$ with $B \setminus A = S$, $s \geq 2$.

Such data C , d , f_k and H_S determine, conversely, a special N -diagram by setting $d_A = d$, $H_{B,A} = H_{B \setminus A}$ if $b - a \geq 2$, and $f_{B,A} = f_{B \setminus A}$ for $b - a = 1$.

A *special N -cube* is a special N -diagram which is also a homotopy commutative N -cube.

As shown in §I.3 a special N -cube has the property that C is a cochain complex with differential d , that the maps $f_k: C \longrightarrow C$ are cochain maps, and that $H_{\{k < \ell\}}$ is a homotopy from $f_\ell \circ f_k$ to $f_k \circ f_\ell$. This was deduced from the equation $D(F) \circ D(F) = 0$ which is the defining property of an N -cube.

Let us now analyse the data specifying higher homotopies in more detail. Let T denote the totalisation of a special N -diagram, with associated map $D = D(F)$. Let $B \supseteq A$ and C be given with $C \cap B = \emptyset$ and $b - a \geq 3$. Then considering the entries (B, A) and $(B \amalg C, A \amalg C)$ of the equation $D \circ D = 0$ we obtain two different conditions involving the map $H_{B \setminus A}$, and they should be consistent in order to make the definition of a special N -cube meaningful. It is clearly enough to do so for C a one element set.

Lemma I.4.2. *Given sets $B \supseteq A$ and an element $z \in N \setminus B$, the two maps*

$$(D \circ D)_{B,A} \quad \text{and} \quad (D \circ D)_{B \amalg z, A \amalg z}$$

agree up to sign.

Proof. For $B \supseteq S \supseteq A$ the summands in (I.2.3) corresponding to S and $S \amalg z$, respectively, agree up to sign since our homotopy commutative N -cube is a special N -cube. So it remains to check that the difference of signs does not depend on S . In view of Lemma I.1.4 it is enough to check that the difference in parity of $(b+1)(s+1) + (s+1)(a+1) = (s+1)(b+a+2)$ and $bs+sa = s(b+a)$ does not depend on s . But if $b+a$ is even both numbers are even, while if $a+b$ is odd the two numbers have different parity, independent of S . \square

Corollary I.4.3. *A cochain complex C with differential d , cochain maps $f_k: C \longrightarrow C$ for $k \in N$, and maps $H_S: C \longrightarrow C$ of graded modules of degree $1 - s$, for $s \geq 2$, determine a special N -cube if and only if for all $S = \{k < \ell\}$ the map $H_{\{k < \ell\}}$ is a homotopy from $f_\ell \circ f_k$ to $f_k \circ f_\ell$, and for every subset $S \subseteq N$, $s \geq 3$, we have*

$$\begin{aligned} 0 = & (-1)^s [S : \emptyset] \cdot d \circ H_S + [S : \emptyset] \cdot H_S \circ d \\ & + \sum_{z \in S} [S : S \setminus z] [S \setminus z : \emptyset] \cdot f_z \circ H_{S \setminus z} \\ & + (-1)^s \sum_{z \in S} [S : z] \cdot H_{S \setminus z} \circ f_z \\ & + \sum_{\substack{T \subseteq S \\ t \geq 2 \leq s-t}} (-1)^{ts} [S : T] [T : \emptyset] \cdot H_{S \setminus T} \circ H_T, \end{aligned}$$

where $t = \#T$ in the last sum (so that T varies over all subsets of S such that both T and $S \setminus T$ have at least two elements).

Proof. Let D be the map associated to the totalisation of the special N -diagram determined by the given data. This diagram is a special N -cube if and only if $D \circ D = 0$, which happens if and only if $(D \circ D)_{B,A} = 0$ for every pair of sets $B \supseteq A$. In view of Lemma I.4.2 this is equivalent to the condition $(D \circ D)_{S,\emptyset} = 0$ for every $S \subseteq N$. Now for $s = 0$ this in turn means that d is a differential, for $s = 1$ this is equivalent to the f_k being cochain maps, and for $s \geq 2$ this is a reformulation of the hypotheses on the maps H_S (bearing formula (I.2.3) in mind). \square

I.5. MAIN EXAMPLES

Commutative diagrams. A cochain complex C of R -modules with differential d together with a collection of pairwise commuting cochain maps $f_k: C \longrightarrow C$ for $k \in N$ defines a special N -cube upon setting $H_S = 0$ for $s \geq 2$. That is, a commutative cubical diagram of self-maps of C can be considered in an obvious way as a homotopy commutative N -cube. We will refer to such a special N -cube as a *trivial* one, and use the symbol $\mathfrak{Triv}(C; f_1, f_2, \dots, f_n)$ to denote the corresponding N -diagram. The totalisation of a trivial cube is in fact the totalisation of a multi-complex, see Proposition III.2.1 below.

Mapping cones. For $N = \{1\}$ a special N -cube is specified by a cochain map $f_1: C \longrightarrow C$. Its totalisation has associated differential

$$D = \begin{pmatrix} d & 0 \\ f_1 & -d \end{pmatrix},$$

and is one version of the mapping cone of f_1 , up to shift. (This differs from other versions in sign and indexing conventions.)

Square diagrams. Slightly more interesting is the case of $N = \{1 < 2\}$, two self-maps $f_1, f_2: C \longrightarrow C$ and a specified homotopy $H = H_N: f_2 \circ f_1 \simeq f_1 \circ f_2$ so that $d \circ H_N - H_N \circ d = f_1 \circ f_2 - f_2 \circ f_1$. This data describes a special N -diagram; the map D associated to its totalisation takes the form (omitting trivial entries, but showing indexing subsets of N)

$$D = \begin{matrix} & \emptyset & \{1\} & \{2\} & N \\ \begin{matrix} \emptyset \\ \{1\} \\ \{2\} \\ N \end{matrix} & \begin{pmatrix} d & & & \\ f_1 & -d & & \\ f_2 & & -d & \\ -H & -f_2 & f_1 & d \end{pmatrix} \end{matrix}$$

which is easily shown to satisfy $D \circ D = 0$ by direct calculation. That is, the given data does in fact specify a special $\{1 < 2\}$ -cube. Up to shift, sign and naming conventions, the totalisation of a $\{1 < 2\}$ -cube is the “mapping 2-torus analogue” discussed by the authors in [HQ14, §III.7].

Derived homotopy commutative cubes. Our most complicated example shows how one can construct (special) homotopy commutative diagrams from strictly commutative ones, roughly speaking by taking a commutative cubical diagram and replacing all occurring cochain complexes by homotopy equivalent ones. The construction is a main ingredient for manipulations of mapping tori later in the paper.

Theorem I.5.1. *Let D be a cochain complex of R -modules with differential d . Let $h_k: D \longrightarrow D$, $k \in N$, be a collection of pairwise commuting cochain maps. Let C be another cochain complex of R -modules, with differential denoted by d as well, and let $\alpha: C \longrightarrow D$ and $\beta: D \longrightarrow C$ be cochain maps. Suppose that G is a homotopy from id_D to $\alpha \circ \beta$ so that $d \circ G + G \circ d = \alpha \circ \beta - \text{id}_D$. Then C , d and the following data define a special N -cube³:*

- $f_k = \beta \circ h_k \circ \alpha$, for $k \in N$;
- $H_{\{k < \ell\}} = \beta \circ (h_k G h_\ell - h_\ell G h_k) \circ \alpha$, for $\{k < \ell\} \subseteq N$;
- $H_S = \beta \circ \sum_{\sigma \in \Sigma(S)} \text{sgn}(\sigma) h_{\sigma(z_1)} G h_{\sigma(z_2)} G h_{\sigma(z_3)} \cdots G h_{\sigma(z_s)} \circ \alpha$, for every set $S = \{z_1 < z_2 < \cdots < z_s\} \subseteq N$ with $s \geq 3$.

Definition I.5.2. The special N -cube defined in the previous Theorem is denoted $\mathfrak{Der}(C; \alpha, \beta, G; h_1, h_2, \dots, h_n)$, and is called the special N -cube derived from the trivial N -cube $\mathfrak{Triv}(D; h_1, h_2, \dots, h_n)$.

Proof of Theorem I.5.1. We verify that the hypotheses of Corollary I.4.3 are satisfied. First note that since G has degree -1 , the maps H_S have degree $1 - s$ as required ($s \geq 2$). For $S = \{k < \ell\}$ we have $s = 2$ and, since α, β and all the h_j are cochain maps,

$$\begin{aligned} d \circ H_S + H_S \circ d &= d \circ \beta(h_k G h_\ell - h_\ell G h_k) \alpha + \beta(h_k G h_\ell - h_\ell G h_k) \alpha \circ d \\ &= \beta(h_k d G h_\ell - h_\ell d G h_k) \alpha + \beta(h_k G d h_\ell - h_\ell G d h_k) \alpha \\ &= \beta h_k (dG + Gd) h_\ell \alpha - \beta h_\ell (dG + Gd) h_k \alpha \\ &= \beta h_k (\alpha \beta - \text{id}) h_\ell \alpha - \beta h_\ell (\alpha \beta - \text{id}) h_k \alpha \\ &= f_k f_\ell - f_\ell f_k \end{aligned}$$

(the last equality holds since $h_k h_\ell = h_\ell h_k$ by hypothesis) so that $H_{\{k < \ell\}}$ is a homotopy from $f_\ell \circ f_k$ to $f_k \circ f_\ell$ as required.

Let us now consider the case $s = 3$, $S = \{z_1 < z_2 < z_3\}$. We can compute incidence numbers:

$$\begin{aligned} [S : \emptyset] &= -1; \\ [S : S \setminus z_j] &= -(-1)^j && \text{for } j = 1, 2, 3; \\ [S : z_j] &= (-1)^j && \text{for } j = 1, 2, 3. \end{aligned}$$

We thus need to verify that the sum

$$(d \circ H_S - H_S \circ d) + \underbrace{\sum_{j=1}^3 (-1)^j \cdot \beta h_{z_j} \alpha \circ H_{S \setminus z_j}}_X - \underbrace{\sum_{j=1}^3 (-1)^j \cdot H_{S \setminus z_j} \circ \beta h_{z_j} \alpha}_Y \quad (\text{I.5.3})$$

³leaving out the composition symbol \circ occasionally, as is standard

is trivial. Now $H_S = \beta \circ \sum_{\sigma \in \Sigma_3} \text{sgn}(\sigma) h_{\sigma(1)} G h_{\sigma(2)} G h_{\sigma(3)} \circ \alpha$ so that⁴

$$\begin{aligned}
d \circ H_S - H_S \circ d &= \beta \circ \sum_{\sigma \in \Sigma_3} \text{sgn}(\sigma) (h_{\sigma(1)} \mathbf{d} G h_{\sigma(2)} G h_{\sigma(3)} - h_{\sigma(1)} G h_{\sigma(2)} G \mathbf{d} h_{\sigma(3)}) \circ \alpha \\
&= \beta \circ \sum_{\sigma \in \Sigma_3} \text{sgn}(\sigma) h_{\sigma(1)} \left(\underbrace{(dG + Gd) h_{\sigma(2)} G}_A - \underbrace{G h_{\sigma(2)} (dG + Gd)}_B \right) h_{\sigma(3)} \circ \alpha.
\end{aligned} \tag{I.5.4}$$

Since $dG + Gd = \alpha\beta - \text{id}$ we get a contribution of

$$\text{sgn}(\sigma) \beta h_{\sigma(1)} (\alpha\beta - \text{id}) h_{\sigma(2)} G h_{\sigma(3)} \alpha \tag{I.5.5}$$

from the term called A in (I.5.4) above. Now note that there are precisely two permutation with $\sigma(3) = j$ a fixed value, with opposite signum; since $h_{\sigma(1)} h_{\sigma(2)} = h_{\sigma(2)} h_{\sigma(1)}$ this means that the factors coming from “id” in (I.5.5) cancel pairwise when summing up over all permutations σ . The remaining $3! = 6$ terms are

$$\begin{aligned}
&\beta(h_2 \alpha \beta h_3 G h_1 - h_3 \alpha \beta h_2 G h_1) \alpha && (\text{for } \sigma(3) = 1) \\
&-\beta(h_1 \alpha \beta h_3 G h_2 - h_3 \alpha \beta h_1 G h_2) \alpha && (\text{for } \sigma(3) = 2) \\
&+\beta(h_1 \alpha \beta h_2 G h_3 - h_2 \alpha \beta h_1 G h_3) \alpha && (\text{for } \sigma(3) = 3)
\end{aligned}$$

while the expanded sum of the expression X in (I.5.3) reads

$$\begin{aligned}
&-\beta h_1 \alpha \beta (h_2 G h_3 - h_3 G h_2) \alpha && (j = 1) \\
&+\beta h_2 \alpha \beta (h_1 G h_3 - h_3 G h_1) \alpha && (j = 2) \\
&-\beta h_3 \alpha \beta (h_1 G h_2 - h_2 G h_1) \alpha. && (j = 3)
\end{aligned}$$

That is, after summing up over all permutations σ the terms coming from A and from X cancel each other.

Similarly, using $dG + Gd = \alpha\beta - \text{id}$ and summing up over all σ gives a non-vanishing contribution of

$$\begin{aligned}
&\beta(h_2 G h_3 \alpha \beta h_1 - h_3 G h_2 \alpha \beta h_1) \alpha && (\text{for } \sigma(3) = 1) \\
&-\beta(h_1 G h_3 \alpha \beta h_2 - h_3 G h_1 \alpha \beta h_2) \alpha && (\text{for } \sigma(3) = 2) \\
&+\beta(h_1 G h_2 \alpha \beta h_3 - h_2 G h_1 \alpha \beta h_3) \alpha && (\text{for } \sigma(3) = 3)
\end{aligned}$$

from B in (I.5.4), and expanding Y from (I.5.3) yields

$$\begin{aligned}
&-\beta(h_2 G h_3 - h_3 G h_2) \alpha \beta h_1 \alpha \\
&+\beta(h_1 G h_3 - h_3 G h_1) \alpha \beta h_2 \alpha \\
&-\beta(h_1 G h_2 - h_2 G h_1) \alpha \beta h_3 \alpha
\end{aligned}$$

cancelling the previous non-trivial contributions from B . This finishes the case $s = 3$.

⁴here and in the sequel we take the liberty to write $\sigma(i)$ in place of $\sigma(z_i)$ and h_i instead of h_{z_i} , to simplify notation

We now turn our attention to the case $s \geq 4$, $S = \{z_1 < z_2 < \dots < z_s\}$. We need to check that the sum

$$(-1)^s [S : \emptyset] \cdot d \circ H_S + [S : \emptyset] \cdot H_S \circ d \quad (\text{I.5.6a})$$

$$+ \sum_{z \in S} [S : S \setminus z] [S \setminus z : \emptyset] \cdot f_z \circ H_{S \setminus z} \quad (\text{I.5.6b})$$

$$+ (-1)^s \sum_{z \in S} [S : z] \cdot H_{S \setminus z} \circ f_z \quad (\text{I.5.6c})$$

$$+ \sum_{\substack{T \subseteq S \\ \bullet \\ t \geq 2 \leq s-t}} (-1)^{ts} [S : T] [T : \emptyset] \cdot H_{S \setminus T} \circ H_T \quad (\text{I.5.6d})$$

(where $t = \#T$ in the last sum) is trivial. Using the definition of H_S in (I.5.6a) and introducing pairwise cancelling terms of the type

$$\begin{aligned} & (-1)^{\ell+1} \beta (h_{\sigma(1)} G h_{\sigma(2)} \dots G \boxed{d} h_{\sigma(\ell)} G h_{\sigma(\ell+1)} \dots G h_{\sigma(n)} \\ & \quad - h_{\sigma(1)} G h_{\sigma(2)} \dots G h_{\sigma(\ell)} \boxed{d} G h_{\sigma(\ell+1)} \dots G h_{\sigma(n)}) \alpha \end{aligned}$$

(where we write h_i instead of h_{z_i} as before) we see that up to the factor $(-1)^s [S : \emptyset]$, (I.5.6a) is the sum over all $\sigma \in \Sigma_s$ of

$$\left. \begin{aligned} & \text{sgn}(\sigma) \cdot \beta \left(h_{\sigma(1)} \boxed{(dG + Gd)} h_{\sigma(2)} G h_{\sigma(3)} G \dots G h_{\sigma(s)} \right. \\ & \quad - h_{\sigma(1)} G h_{\sigma(2)} \boxed{(dG + Gd)} h_{\sigma(3)} G \dots G h_{\sigma(s)} \\ & \quad + h_{\sigma(1)} G h_{\sigma(2)} G h_{\sigma(3)} \boxed{(dG + Gd)} \dots G h_{\sigma(s)} \\ & \quad \quad \quad \ddots \\ & \quad \left. + (-1)^s h_{\sigma(1)} G h_{\sigma(2)} G h_{\sigma(3)} G \dots h_{\sigma(s-1)} \boxed{(dG + Gd)} h_{\sigma(s)} \right) \alpha . \end{aligned} \right\} \quad (\text{I.5.7})$$

Now $dG + Gd = \alpha\beta - \text{id}$; since the maps h_j commute, all contributions from “id” will cancel each other upon summing up over σ . More precisely, by multiplying out the ℓ th summand of (I.5.7) becomes, up to the sign $(-1)^{\ell+1} \cdot \text{sgn}(\sigma)$,

$$\beta h_{\sigma(1)} G h_{\sigma(2)} G \dots \boxed{h_{\sigma(\ell)} \alpha \beta h_{\sigma(\ell+1)}} \dots G h_{\sigma(s)} \alpha \quad (\text{I.5.7}_{\alpha\beta})$$

$$- \beta h_{\sigma(1)} G h_{\sigma(2)} G \dots \boxed{h_{\sigma(\ell)} h_{\sigma(\ell+1)}} \dots G h_{\sigma(s)} \alpha , \quad (\text{I.5.7}_{\text{id}})$$

and the summand (I.5.7_{id}) cancels out with the corresponding one coming from the permutation $\sigma \circ (\ell, \ell + 1)$.

So we are left with the following non-trivial contribution from (I.5.7), for a fixed permutation σ :

$$(-1)^s [S : \emptyset] \cdot \text{sgn}(\sigma) \beta \left(h_{\sigma(1)} \alpha \beta h_{\sigma(2)} G h_{\sigma(3)} G \dots h_{\sigma(s-1)} G h_{\sigma(s)} \right. \quad (\text{I.5.7}_1)$$

$$\left. - h_{\sigma(1)} G h_{\sigma(2)} \alpha \beta h_{\sigma(3)} G \dots h_{\sigma(s-1)} G h_{\sigma(s)} \right. \quad (\text{I.5.7}_2)$$

$$\left. + h_{\sigma(1)} G h_{\sigma(2)} G h_{\sigma(3)} \alpha \beta \dots h_{\sigma(s-1)} G h_{\sigma(s)} \right. \quad (\text{I.5.7}_3)$$

$$\vdots$$

$$\left. + (-1)^s h_{\sigma(1)} G h_{\sigma(2)} G h_{\sigma(3)} G \dots h_{\sigma(s-1)} \alpha \beta h_{\sigma(s)} \right) \alpha \quad (\text{I.5.7}_{s-1})$$

We will proceed by expanding the summands (I.5.6b–d) and show that there is a bijective, sign-reversing correspondence of the resulting terms with the terms (I.5.7₁–I.5.7_{s-1}) just calculated.

Let us begin with (I.5.6b). The maps $H_{S \setminus z}$ are given as sums of $(s-1)!$ terms indexed by the permutations of $S \setminus z$, while z itself varies over the set S ; expanding results in $s \cdot (s-1)! = s!$ terms. More explicitly, writing $S = \{z_1 < z_2 < \dots < z_s\}$ these $s \cdot (s-1)!$ summands are displayed as

$$\sum_{k=1}^s \sum_{\tau \in \Sigma(S \setminus z_k)} [S : S \setminus z_k] [S \setminus z_k : \emptyset] \cdot \text{sgn}(\tau) \cdot \beta h_k \alpha \beta h_{\tau(1)} G h_{\tau(2)} G \dots G h_{\tau(k-1)} G h_{\tau(k+1)} G \dots G h_{\tau(s)} \alpha ,$$

writing h_i for h_{z_i} and $\tau(i)$ for $\tau(z_i)$ as before. Individually they will correspond, in a bijective manner, to the $s!$ different summands (I.5.7₁), which are indexed by the permutation σ .

In detail, given a permutation $\sigma \in \Sigma(S)$ let k be the index determined by $z_k = \sigma(z_1)$. Let $\tau \in \Sigma(S \setminus z_k)$ be defined by

$$\tau(z_i) = \begin{cases} \sigma(z_{i+1}) & \text{for } i < k, \\ \sigma(z_i) & \text{for } i > k. \end{cases} \quad (\text{I.5.8})$$

Then by construction of τ we have an equality of s -tuples

$$\begin{aligned} & (\sigma(z_1), \sigma(z_2), \dots, \sigma(z_s)) \\ &= (z_k, \tau(z_1), \tau(z_2), \dots, \tau(z_{k-1}), \tau(z_{k+1}), \dots, \tau(z_s)) \end{aligned}$$

so that the summand of (I.5.6b) indexed by $z = z_k$ and $\tau \in S \setminus z$ agrees with (I.5.7₁) up to sign, and it remains to show that the signs are different.

Now the sign occurring in (I.5.6b) is $[S : S \setminus z_k] [S \setminus z_k : \emptyset] \cdot \text{sgn}(\tau)$. Let us relate $\text{sgn}(\tau)$ to $\text{sgn}(\sigma)$ now. The set of inversions⁵ of τ is

$$\text{Inv}(\tau) = \{(z_i, z_j) \mid i, j \neq k; i < j; \tau(z_i) > \tau(z_j)\}.$$

Similarly, the set of inversions of σ is given by

$$\text{Inv}(\sigma) = \{(z_i, z_j) \mid i < j; \sigma(z_i) > \sigma(z_j)\},$$

⁵an inversion of a permutation π of a totally ordered finite set is a pair of elements (x, y) of its domain with $x < y$ and $\pi(x) > \pi(y)$

and the former injects into the latter by a map induced from the assignment $t \mapsto t + 1$ if $t < k$, and $t \mapsto t$ if $t > k$. The inversions of σ not in the image of this injection are precisely the inversions of the type (z_1, z_j) , which are parametrised by those $j > 1$ satisfying $\sigma(z_j) < \sigma(z_1) = z_k$. But as j varies over all integers between 2 and s , the image $\sigma(z_j)$ varies over all of $S \setminus z_k$ so that the number of such inversions is the number of elements of S strictly less than z_k , of which there are $k - 1$. In total, $\#\text{Inv}(\sigma) = \#\text{Inv}(\tau) + k - 1$. We thus have

$$\begin{aligned} \text{sgn}(\sigma) &= (-1)^{\text{Inv}(\sigma)} \\ &= (-1)^{\text{Inv}(\tau)} \cdot (-1)^{k-1} \\ &= \text{sgn}(\tau) \cdot [S : S \setminus z_k] , \end{aligned} \tag{I.5.9}$$

the last equality courtesy of Lemma I.1.3. Thus the total sign occurring in (I.5.6b) is

$$\begin{aligned} [S : S \setminus z_k][S \setminus z_k : \emptyset] \cdot \text{sgn}(\tau) &= [S \setminus z_k : \emptyset] \cdot \text{sgn}(\sigma) \\ &= (-1)^{(s-1)(s-2)/2} \cdot \text{sgn}(\sigma) \end{aligned}$$

while the sign occurring in (I.5.7₁) is

$$(-1)^s [S : \emptyset] \cdot \text{sgn}(\sigma) = (-1)^s \cdot (-1)^{s(s-1)/2} \cdot \text{sgn}(\sigma) = (-1)^{s(s+1)/2} \cdot \text{sgn}(\sigma) .$$

It remains to observe that

$$\frac{(s-1)(s-2)}{2} = \frac{s^2}{2} - \frac{3s}{2} + 1 \quad \text{and} \quad \frac{s^2}{2} + \frac{s}{2} = \frac{s(s+1)}{2}$$

differ by $2s - 1$ and thus have different parity. This implies that

$$[S : S \setminus z_k][S \setminus z_k : \emptyset] \cdot \text{sgn}(\tau) = -((-1)^s [S : \emptyset] \cdot \text{sgn}(\sigma))$$

as required.

Not surprisingly, a similar argument works for (I.5.6c) which is linked to the term (I.5.7_{s-1}). Write $S = \{z_1 < z_2 < \dots < z_s\}$ as usual; expanding (I.5.6c) results in the $s \cdot (s-1)!$ -term sum

$$\begin{aligned} \sum_{k=1}^s \sum_{\tau \in \Sigma(S \setminus z_k)} (-1)^s \cdot [S : z_k] \cdot \text{sgn}(\tau) \\ \cdot \beta h_{\tau(1)} G h_{\tau(2)} G \dots G h_{\tau(k-1)} G h_{\tau(k+1)} G \dots G h_{\tau(s)} \alpha \beta h_k \alpha , \end{aligned}$$

again writing h_i for h_{z_i} and $\tau(i)$ for $\tau(z_i)$. The summand indexed by k and $\tau \in \Sigma(S \setminus z_k)$ is uniquely determined by $\sigma \in \Sigma(S)$ as follows: k is the index satisfying $z_k = \sigma(z_s)$, and τ is given by

$$\tau(z_i) = \begin{cases} \sigma(z_i) & \text{for } i < k , \\ \sigma(z_{i-1}) & \text{for } i > k . \end{cases}$$

Then by construction of τ we have an equality of s -tuples

$$\begin{aligned} (\sigma(z_1), \sigma(z_2), \dots, \sigma(z_s)) \\ = (\tau(z_1), \tau(z_2), \dots, \tau(z_{k-1}), \tau(z_{k+1}), \dots, \tau(z_s), z_k) \end{aligned}$$

so that the summand of (I.5.6c) indexed by $z = z_k$ and $\tau \in S \setminus z$ agrees with (I.5.7_{s-1}) up to sign, and it remains to show that the signs are different.

There is an injective map of sets of inversions

$$\text{Inv}(\tau) \longrightarrow \text{Inv}(\sigma)$$

induced by the assignment $t \mapsto t$ if $t < k$, and $t \mapsto t - 1$ if $t > k$. The inversions of σ not in the image are of the type (i, s) where $\sigma(z_i) > \sigma(z_s) = z_k$. But as i varies over all numbers from 1 to $s - 1$, the element $\sigma(z_i)$ varies over all of $S \setminus z_k$. That is, the number of inversions of σ which are not in the image is the number of elements of S which are strictly greater than z_k , of which there are $s - k$ many. In other words,

$$\text{sgn}(\sigma) = \text{sgn}(\tau) \cdot (-1)^{s-k}.$$

So the sign occurring in (I.5.7_{s-1}) is

$$\begin{aligned} (-1)^s [S : \emptyset] (-1)^s \cdot \text{sgn}(\sigma) &= (-1)^{s(s-1)/2} \cdot \text{sgn}(\sigma) \\ &= (-1)^{s(s-1)/2} \cdot (-1)^{s-k} \cdot \text{sgn}(\tau) \\ &= (-1)^{s(s+1)/2-k} \cdot \text{sgn}(\tau). \end{aligned}$$

On the other hand, by Lemma I.1.3 we know that $[S : z_k] = (-1)^\kappa$, where κ is the number of pairs $(z_i, z_j) \in S \times S$ with $i < j$ and $j \neq k$; that is, κ is the number of unordered pairs of elements of S , reduced by the number of elements in S strictly less than z_k :

$$\kappa = \frac{1}{2}s(s-1) - (k-1)$$

This means that the sign occurring in (I.5.6c) is

$$\begin{aligned} (-1)^s \cdot [S : z_k] \cdot \text{sgn}(\tau) &= (-1)^s \cdot (-1)^{s(s-1)/2-k+1} \cdot \text{sgn}(\tau) \\ &= -(-1)^{s(s+1)/2-k} \cdot \text{sgn}(\tau), \end{aligned}$$

which is opposite to the above as required.

It remains to deal with (I.5.7_ℓ), for $1 < \ell < s - 1$, and relate these to the summands of (I.5.6d). For reference, we record that the latter expands to

$$\left. \begin{aligned} \sum_{\substack{T \subseteq S \\ t \geq 2 \leq s-t}} \sum_{\tau \in \Sigma(S \setminus T)} \sum_{\mu \in \Sigma(T)} (-1)^{ts} [S : T] [T : \emptyset] \cdot \text{sgn}(\tau) \cdot \text{sgn}(\mu) \\ \cdot \beta h_{\tau(1)} G h_{\tau(2)} G \dots G h_{\tau(\ell)} \alpha \beta h_{\mu(1)} G h_{\mu(2)} G \dots G h_{\mu(s-\ell)} \alpha, \end{aligned} \right\} \quad (\text{I.5.10})$$

where we have written $\ell = s - t$, $S \setminus T = \{x_1 < x_2 < \dots < x_\ell\}$ and $T = \{y_1 < y_2 < \dots < y_{s-\ell}\}$, and also used the abbreviations $\tau(i) = \tau(x_i)$ and $\mu(j) = \mu(y_j)$.

For $S = \{z_1 < \dots < z_s\}$ as usual, define $T = \{\sigma(\ell+1), \sigma(\ell+2), \dots, \sigma(s)\}$ so that $t = \#T = s - \ell$. Rename the elements of $S \setminus T = \{\sigma(1), \dots, \sigma(\ell)\}$ as $\{x_1 < x_2 < \dots < x_\ell\}$, and let $\tau \in \Sigma(S \setminus T)$ be determined by $\tau(x_j) = \sigma(z_j)$ for $1 \leq j \leq \ell$. That is, τ is the unique permutation of $S \setminus T$ satisfying

$$\tau^{-1}\sigma(z_1) < \tau^{-1}\sigma(z_2) < \dots < \tau^{-1}\sigma(z_\ell). \quad (\text{I.5.11})$$

Similarly, let $\mu \in \Sigma(T)$ be the unique permutation with

$$\mu^{-1}\sigma(z_{\ell+1}) < \mu^{-1}\sigma(z_{\ell+2}) < \dots < \mu^{-1}\sigma(z_s). \quad (\text{I.5.12})$$

By construction the summand of (I.5.6d) indexed by T , τ and μ agrees with (I.5.7 $_\ell$) up to sign, and this correspondence is uniquely reversible.

It remains to check that the signs are different. To this end, let $\nu = (\tau^{-1}, \mu^{-1}) \circ \sigma \in \Sigma(S)$ where we use the canonical embedding

$$(\tau^{-1}, \mu^{-1}) \in \Sigma(S \setminus T) \times \Sigma(T) \subseteq \Sigma(S)$$

coming from the decomposition $S = (S \setminus T) \amalg T$. Then on the one hand,

$$\text{sgn}(\nu) = \text{sgn}(\tau) \cdot \text{sgn}(\mu) \cdot \text{sgn}(\sigma) . \quad (\text{I.5.13})$$

On the other hand, in view of (I.5.11) and (I.5.12) we know that the inversions of ν are parametrised by the pairs (i, j) with $i \leq \ell < j$ and $\nu(z_i) > \nu(z_j)$. Now the elements of the form $\nu(i)$, for $1 \leq i \leq \ell$, are precisely the elements of the form $\sigma(i)$, for i in the same range, and are thus precisely the elements of $S \setminus T$; similarly, the elements of the form $\nu(j)$, for $\ell < i \leq s$ are precisely the elements of T . In other words, the number λ of inversions of ν is the number of pairs $(y, x) \in T \times (S \setminus T)$ with $y < x$.

Let ω be the number of pairs $(y, x) \in (S \setminus T) \times (S \setminus T)$ with $y < x$. Then

$$\omega = \frac{1}{2} \ell(\ell - 1) ,$$

by direct counting. Clearly $\kappa = \lambda + \omega$ is the number of pairs $(y, x) \in S \times (S \setminus T)$ with $y < x$. But $(-1)^\kappa = [S : T]$, by Lemma I.1.3; combined with (I.5.13) this yields

$$\begin{aligned} [S : T] &= (-1)^\omega \cdot (-1)^\lambda = (-1)^{\ell(\ell-1)/2} \cdot \text{sgn}(\nu) \\ &= (-1)^{\ell(\ell-1)/2} \cdot \text{sgn}(\tau) \cdot \text{sgn}(\mu) \cdot \text{sgn}(\sigma) . \end{aligned}$$

So the sign occurring in (I.5.6d), or rather in its expanded form (I.5.10), is given by

$$\begin{aligned} &(-1)^{ts} [S : T] [T : \emptyset] \cdot \text{sgn}(\tau) \cdot \text{sgn}(\mu) \\ &= (-1)^{(s-\ell)s} \cdot (-1)^{\ell(\ell-1)/2} [T : \emptyset] \cdot \text{sgn}(\sigma) \\ &= (-1)^{(s-\ell)s} \cdot (-1)^{\ell(\ell-1)/2} [T : \emptyset] \cdot (-1)^{(s-\ell)(s-\ell-1)/2} \cdot \text{sgn}(\sigma) , \end{aligned}$$

and a calculation we omit shows that this agrees with

$$(-1)^{\ell+s(s+1)/2} \cdot \text{sgn}(\sigma) .$$

The sign occurring in (I.5.7 $_\ell$) is the opposite as

$$\begin{aligned} (-1)^s [S : \emptyset] \cdot \text{sgn}(\sigma) \cdot (-1)^{\ell+1} &= (-1)^s \cdot (-1)^{s(s-1)/2} \cdot (-1)^{\ell+1} \cdot \text{sgn}(\sigma) \\ &= -(-1)^{\ell+s(s+1)/2} \cdot \text{sgn}(\sigma) , \end{aligned}$$

just as required. This finishes the proof of Theorem I.5.1. \square

Part II. Mapping tori and the Mather trick

II.1. FILTRATION

We now record that the totalisation construction introduced in Definition I.2.2 comes with a natural filtration by cardinality of the index set.

That is, given a homotopy commutative N -cube F , cf. Definition I.3.1, define for fixed $k \in \mathbb{Z}$ the modules

$$\mathrm{Tot}_k(F)^n := \bigoplus_{\substack{A \subseteq N \\ \#A \geq k}} F(A)^{n-a}$$

together with module homomorphisms

$$D(F)_k = D_k = (D_{k,B,A})_{A,B \subseteq N} : \mathrm{Tot}_k(F)^n \longrightarrow \mathrm{Tot}_k(F)^{n+1}$$

given by

$$D_{k,B,A} = \begin{cases} (-1)^{ab} [B : A] \cdot H_{B,A} & \text{if } A \subseteq B \text{ and } \#A \geq k, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition II.1.1. *The graded module $\mathrm{Tot}_k(F)$ together with $D(F)_k$ is a cochain complex. We have a descending filtration of subcomplexes*

$$\mathrm{Tot}(F) = \mathrm{Tot}_0(F) \supseteq \mathrm{Tot}_1(F) \supseteq \dots \supseteq \mathrm{Tot}_n(F) \supseteq \mathrm{Tot}_{n+1}(F) = 0.$$

The filtration quotients are

$$\mathrm{Tot}_k(F) / \mathrm{Tot}_{k+1}(F) = \bigoplus_{\substack{A \subseteq N \\ \#A = k}} \Sigma^k F(A),$$

that is, are the direct sums of the cochain complexes $F(A)$ with $\#A = k$, suspended⁶ k times. \square

II.2. THE MATHER TRICK FOR DERIVED SPECIAL N -CUBES

Let h_1, \dots, h_n be pairwise commuting self-maps of the cochain complex D . As mentioned at the beginning of §I.5, we can form the trivial special N -cube $\mathfrak{Triv}(D; h_1, h_2, \dots, h_n)$ (with trivial homotopies and higher homotopies); for convenience we denote its totalisation by X and its differential by $D^X = (D_{B,A}^X)_{A,B \subseteq N}$.

Let $g: D \longrightarrow D$ be a cochain map, and let $G: \mathrm{id} \simeq g$ be a homotopy so that $dG + Gd = g - \mathrm{id}$. We can now form the derived special N -cube $\mathfrak{Der}(D; g, \mathrm{id}_D, G; h_1, h_2, \dots, h_n)$ according to Theorem I.5.1 and Definition I.5.2, with $\beta = \mathrm{id}_D$ and $\alpha = g$. We retain the notation of I.5.1: we have chain maps $f_k = h_k g$ and higher homotopies H_s (for $s \geq 2$), and maps $H_{B,A} = H_{B \setminus A}$. We denote the totalisation by Y , and write $D^Y = (D_{B,A}^Y)_{A,B \subseteq N}$ for its differential.

Finally, we define a matrix $M = (M_{B,A})_{A,B \subseteq N}$ which, *a priori*, is just a collection of module homomorphisms $Y^n \longrightarrow X^n$. We set $M_{B,A} = 0$ if $A \not\subseteq B$. Otherwise, we set

$$M_{B,A} = (-1)^b (-1)^{ab} [B : A] \cdot M_{B \setminus A}, \quad (\text{II.2.1})$$

where $a = \#A$ and $b = \#B$ as usual, and

$$\left. \begin{aligned} M_\emptyset &= g, \\ M_{\{k\}} &= G \circ f_k = G \circ h_k \circ g \quad \text{for } k \in S, \\ M_S &= G \circ H_S \quad \text{for } s \geq 2. \end{aligned} \right\} \quad (\text{II.2.2})$$

⁶The k th suspension of cochain complexes modifies the differential by a factor of $(-1)^k$.

Writing $S = \{z_1 < z_2 < \dots < z_s\}$ and abbreviating h_{z_i} by h_i as before we have more explicitly, for $s \geq 2$,

$$M_S = G \circ \sum_{\sigma \in \Sigma_s} \text{sgn}(\sigma) \cdot h_{\sigma(1)} G h_{\sigma(2)} G \dots G h_{\sigma(s)} \circ g .$$

Lemma II.2.3. *The matrix M defines a cochain map*

$$Y = \text{Tot} \mathfrak{D}\mathfrak{r}\mathfrak{t}(D; g, \text{id}_D, G; h_1, h_2, \dots, h_n) \xrightarrow{M} \text{Tot} \mathfrak{T}\mathfrak{r}\mathfrak{i}\mathfrak{v}(D; h_1, h_2, \dots, h_n) = X .$$

This map is a quasi-isomorphism. If D is a bounded above complex of projective R -modules, then M is a homotopy equivalence $M: Y \simeq X$.

Proof. Granting that M defines a cochain map, it is almost trivial to verify that M respects the filtration of the totalisation, such that the induced map on filtration quotients

$$Y_k/Y_{k+1} \longrightarrow X_k/X_{k+1}$$

is a direct sum of the (k th suspension of) the cochain map g . This implies, by an iterative application of the five lemma for $k = n, n-1, \dots, 0$, that $M: Y_k \longrightarrow X_k$ is a quasi-isomorphism. In particular, M is a quasi-isomorphism $Y = Y_0 \longrightarrow X_0 = X$. The final claim about bounded above complexes follows from general homological algebra.

We now check that M defines a cochain map; to this end we need to verify that $MD^Y = D^X M$, that is, using the fact that all matrices under consideration are triangular,

$$(MD^Y)_{B,A} - (D^X M)_{B,A} = 0$$

for all $A \subseteq B \subseteq N$.

We will analyse the second summand $(D^X M)_{B,A} = \sum_{B \supseteq \dot{S} \supseteq A} D_{B,S}^X M_{S,A}$ first, and assume that $b-a \geq 3$. As X is the totalisation of a trivial N -cube we know that $D_{B,S}^X = 0$ unless $\#(B \setminus S) \leq 1$; this means that

$$\begin{aligned} (D^X M)_{B,A} &= (-1)^b d \cdot (-1)^b (-1)^{ab} [B : A] \cdot G \circ H_{B \setminus A} \\ &\quad + \sum_{z \in B \setminus A} [B : B \setminus z] \cdot h_z \circ (-1)^{b-1} (-1)^{a(b-1)} [B \setminus z : A] \cdot G \circ H_{(B \setminus z) \setminus A} \end{aligned}$$

so that, simplifying the sign terms and introducing a term of the form $GdH_{B \setminus A} - GdH_{B \setminus A}$, we obtain

$$\begin{aligned} (-1)^{ab} (D^X M)_{B,A} &= [B : A] ((dG + Gd) \circ H_{B \setminus A} - GdH_{B \setminus A}) \\ &\quad - (-1)^{b-a} \cdot \sum_{z \in B \setminus A} [B : B \setminus z] [B \setminus z : A] \cdot h_z \circ G \circ H_{(B \setminus z) \setminus A} . \end{aligned}$$

The second line of this expression reduces to $[B : A] \cdot H_{B \setminus A}$, as we will verify presently; as $dG + Gd = g - \text{id}$ this means that

$$(D^X M)_{B,A} = (-1)^{ab} [B : A] \cdot (gH_{B \setminus A} - GdH_{B \setminus A}) . \quad (\text{II.2.4})$$

To verify the claim, let us use the explicit definition of $H_{(B \setminus z) \setminus A}$; writing $B \setminus A = z_1 < z_2 < \dots < z_{b-a}$ and abbreviating h_{z_k} by h_k as usual, we have

$$\begin{aligned} & - \sum_{z \in B \setminus A} [B : B \setminus z][B \setminus z : A] \cdot h_z \circ G \circ H_{(B \setminus z) \setminus A} \\ &= \sum_{k=1}^{b-a} \sum_{\tau \in \Sigma(B \setminus z_k)} -[B : B \setminus z_k][B \setminus z_k : A] \cdot \text{sgn}(\tau) \\ & \quad \cdot h_k G h_{\tau(1)} G h_{\tau(2)} G \dots G h_{\tau(k-1)} G h_{\tau(k+1)} G \dots G h_{\tau(b)} g , \end{aligned}$$

using $\tau(i)$ for $\tau(z_i)$ as before; we need to compare that with

$$[B : A] \cdot H_{B \setminus A} = \sum_{\sigma \in \Sigma(B \setminus A)} [B : A] \cdot \text{sgn}(\sigma) \cdot h_{\sigma(1)} G h_{\sigma(2)} G \dots G h_{\sigma(b)} g .$$

As we have seen before, there is a bijection

$$\Sigma(B \setminus A) \ni \sigma \mapsto (k, \tau) \text{ where } z_k = \sigma(z_1) \in B \text{ and } \tau \in \Sigma((B \setminus z_k) \setminus A)$$

such that $\text{sgn}(\sigma) = (-1)^{k-1} \text{sgn}(\tau)$, see (I.5.8) and (I.5.9). So we have reduced to showing

$$[B : B \setminus z_k][B \setminus z_k : A] = (-1)^{b-a} (-1)^k [B : A] .$$

Now by Lemma I.1.3 the left-hand side is given by $(-1)^\alpha \cdot (-1)^\beta$ where

$$\begin{aligned} \alpha &= \#\{x \in B \mid x < z_k\} , \\ \beta &= \#\{x < y \mid x, y \in B \setminus z_k, x < y, y \notin A\} . \end{aligned}$$

The (disjoint) union of the two sets has the same number of elements as

$$\begin{aligned} & \{x < y \mid x, y \notin A\} \amalg \{x < y \mid x \in A, y \notin A\} \setminus \{y \mid z_k < y, y \notin A\} \\ &= \{x < y \mid x \in B, y \in B \setminus A\} \setminus \{y \mid z_k < y, y \notin A\} ; \end{aligned}$$

the set taken away has clearly $b - a - k$ elements so that, using Lemma I.1.3 again, we have

$$\begin{aligned} [B : B \setminus z_k][B \setminus z_k : A] &= (-1)^\alpha \cdot (-1)^\beta \\ &= [B : A] (-1)^{b-a-k} = (-1)^{b-a} (-1)^k [B : A] \end{aligned}$$

just as required.

We now need to consider the explicit form of

$$(MD^Y)_{B,A} = \sum_{B \supseteq S \supseteq A} M_{B,S} D_{S,A}^Y ;$$

plugging in all relevant definitions, and using (II.2.4), we see that the difference $(MD^Y)_{B,A} - (D^X M)_{B,A}$ is given by the expression

$$\begin{aligned}
& g \circ (-1)^{ab}[B : A] \cdot H_{B \setminus A} \\
& + \sum_{z \in B \setminus A} (-1)^b[B : B \setminus z] \cdot Gf_z \circ (-1)^{(b-1)a}[B \setminus z : A] \cdot H_{(B \setminus z) \setminus A} \\
& + \sum_{\substack{B \supseteq T \supseteq A \\ b-t \geq 2 \leq t-a}} (-1)^b(-1)^{bt}[B : T] \cdot GH_{B \setminus T} \circ (-1)^{at}[T : A] \cdot H_{T \setminus A} \\
& + \sum_{z \in B \setminus A} (-1)^b(-1)^{(a+1)b}[B : A \amalg z] \cdot GH_{B \setminus (A \amalg z)} \circ [A \amalg z : A] \cdot f_z \\
& \quad + (-1)^b(-1)^{ab}[B : A]GH_{B \setminus A} \circ (-1)^ad \\
& \quad - (-1)^{ab}[B : A] \cdot (gH_{B \setminus A} - GdH_{B \setminus A}) . \quad (\text{II.2.5})
\end{aligned}$$

To analyse the sign in the third line of this expression, the sum over T , note that the total exponent of -1 is

$$b + bt + at \equiv b + (t-a)(b-a) + a(b-a) \equiv (b-a) + ab + (t-a)(b-a)$$

modulo 2. That is, $(-1)^{ab}(-1)^{b-a} \cdot ((MD^Y)_{B,A} - (D^X M)_{B,A})$ equals

$$\begin{aligned}
& (-1)^{b-a}[B : A] \cdot g \circ H_{B \setminus A} \\
& + \sum_{z \in B \setminus A} [B : B \setminus z][B \setminus z : A] \cdot G \circ f_z \circ H_{(B \setminus z) \setminus A} \\
& + \sum_{\substack{B \supseteq T \supseteq A \\ b-t \geq 2 \leq t-a}} (-1)^{(t-a)(b-a)}[B : T][T : A] \cdot G \circ H_{B \setminus T} \circ H_{T \setminus A} \\
& + (-1)^{b-a} \cdot \sum_{z \in B \setminus A} [B : A \amalg z][A \amalg z : A] \cdot G \circ H_{B \setminus (A \amalg z)} \circ f_z \\
& \quad + [B : A] \cdot G \circ H_{B \setminus A} \circ d \\
& \quad + (-1)^{b-a}[B : A] \cdot (GdH_{B \setminus A} - gH_{B \setminus A}) .
\end{aligned}$$

We note that the very first summand and the very last (after multiplying out the last pair of parentheses) cancel. We re-write the remaining terms, setting $S = B \setminus A$ so that $s = b - a$. We also let T be a subset of S so that the role of T above is now played by $T \amalg A$, and t has to be replaced by $t + a$. In addition we take out a common post-composition with G . All

told, we now want to show that the following expression is trivial:

$$\begin{aligned}
& \sum_{z \in S} [B : B \setminus z][B \setminus z : A] \cdot f_z \circ H_{S \setminus z} \\
& + \sum_{\substack{T \subseteq S \\ s-t \geq 2 \leq s}} (-1)^{st} [B : A \amalg T][A \amalg T : A] \cdot H_{S \setminus T} \circ H_T \\
& + (-1)^s \cdot \sum_{z \in S} [B : A \amalg z][A \amalg z : A] \cdot H_{S \setminus z} \circ f_z \\
& + [B : A] \cdot H_S \circ d \\
& + (-1)^s [B : A] \cdot d \circ H_S
\end{aligned}$$

Now compare this expression with the analogous one obtained by removing an element $x \in A$ from both A and B , that is, obtained by the substitutions $A \mapsto A \setminus x$ and $B \mapsto B \setminus x$. The numbers t and $s = b - a = (b - 1) - (a - 1)$, and the difference set $S = B \setminus A = (B \setminus x) \setminus (A \setminus x)$ are clearly unaffected by this change. All the products of incidence numbers, as well as the incidence number $[B : A] = [B : B][B : A]$ acquire a sign that depends on A , B and x only, by Lemma I.1.4, so is the same for all summands. That is, up to sign the whole expression remains unchanged, so we may just as well consider the case $A = \emptyset$ only. But then the expression is trivial by Corollary I.4.3.

The verification for $b - a < 3$ can be done along similar lines, but is slightly easier due to fewer terms being involved. We omit the details. \square

We introduce yet more notation: suppose we are given, in addition to the maps h_i above, cochain maps $\alpha : C \longrightarrow D$ and $\beta : D \longrightarrow C$. We set $g = \alpha \circ \beta : D \longrightarrow D$, and let G as before be a homotopy $G : \text{id}_D \simeq \alpha \circ \beta = g$ so that $dG + Gd = \alpha\beta - \text{id}$; this is, according to Theorem I.5.1, precisely the data required to define the special N -cube $\mathfrak{Det}(C; \alpha, \beta, G; h_1, h_2, \dots, h_n)$. We denote its totalisation by Z , and the differential by $D^Z = (D_{B,A}^Z)_{A,B \subseteq N}$.

We let L denote the (constant) diagonal matrix with entry β ; this is to be considered as a module homomorphism $Y^n \longrightarrow Z^n$.

Lemma II.2.6 (MATHER trick for derived special N -cubes). *The matrix L defines a cochain map*

$$\begin{aligned}
& \text{Tot} \mathfrak{Det}(D; \alpha\beta, \text{id}_D, G; h_1, h_2, \dots, h_n) = Y \\
& \xrightarrow{L} Z = \text{Tot} \mathfrak{Det}(C; \alpha, \beta, G; h_1, h_2, \dots, h_n) .
\end{aligned}$$

If β is a quasi-isomorphism, then L is a quasi-isomorphism as well. If in addition C and D are bounded above complexes of projective R -modules, then L is a homotopy equivalence $L : Y \simeq Z$.

Proof. It is a straightforward calculation that $LD^Y = D^Z L$ so that L defines a cochain map. The map L respects the filtration of totalisation, such that the induced map on filtration quotients

$$Y_k / Y_{k+1} \longrightarrow Z_k / Z_{k+1}$$

is a direct sum of the (k th suspension of) the cochain map β . This implies, by an iterative application of the five lemma for $k = n, n-1, \dots, 0$,

that $L: Y_k \longrightarrow Z_k$ is a quasi-isomorphism. In particular, L is a quasi-isomorphism $Y = Y_0 \longrightarrow Z_0 = Z$. The final claim about bounded above complexes follows from general homological algebra. \square

A more explicit treatment of these results (Lemmas II.2.3 and II.2.6) in the special case $n = 2$ of two LAURENT variables can be found in [HQ14, Lemma III.7.4], with slightly different sign conventions.

II.3. HIGHER-DIMENSIONAL MAPPING TORI

Definition II.3.1. Let F be a special N -cube on the R -module cochain complex C ; we think of this as a collection of cochain maps f_k which commute up to specified coherent homotopy. Write $L = R[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$. We define the *mapping n -torus* of F , denoted $\mathcal{T}F$, to be the totalisation of the special N -cube \bar{F} on the L -module complex $\bar{C} = C \otimes_R L$ specified by the following data:

- differential $\bar{d} = d_{\bar{C}} = d_C \otimes 1$;
- $\bar{f}_k = f_k \otimes 1 - 1 \otimes x_k$, for $k \in N$;
- $\bar{H}_S = H_S \otimes 1$, for any $S \subseteq N$ with $s \geq 2$.

Example II.3.2. For pairwise commuting self-maps h_k of C , we have

$$\begin{aligned} \mathcal{T}\mathfrak{T}\mathfrak{riv}(C; h_1, h_2, \dots, h_n) &= \text{Tot}\mathfrak{T}\mathfrak{riv}(C \otimes_R L; h_1 \otimes 1 - 1 \otimes x_1, \\ &\quad h_2 \otimes 1 - 1 \otimes x_2, \dots, h_n \otimes 1 - 1 \otimes x_n) . \end{aligned}$$

Example II.3.3. Let $f_0, f_1: C \longrightarrow C$ be two cochain maps, and let $H = H_{\{0,1\}}: f_1 f_0 \simeq f_0 f_1$ be a given homotopy. These data define a special $\{0 < 1\}$ -cube F , and up to shift, sign and naming conventions the totalisation of the associated special $\{0 < 1\}$ -cube \bar{F} is the mapping 2-torus $\mathcal{T}(f_1, f_0; H)$ as discussed by the authors in [HQ14, §III.6].

Of course we need to check that the definition of mapping tori makes sense:

Lemma II.3.4. *The data listed in Definition II.3.1 define a special N -cube.*

Proof. We verify that the hypotheses of Corollary I.4.3 are satisfied. For $k < \ell$ we have

$$\begin{aligned} \bar{d}\bar{H}_{\{k<\ell\}} + \bar{H}_{\{k<\ell\}}\bar{d} &= (dH_{\{k<\ell\}} + H_{\{k<\ell\}}d) \otimes 1 \\ &= (f_k f_\ell - f_\ell f_k) \otimes 1 \\ &= (f_k \otimes 1 - 1 \otimes x_k)(f_\ell \otimes 1 - 1 \otimes x_\ell) \\ &\quad - (f_\ell \otimes 1 - 1 \otimes x_\ell)(f_k \otimes 1 - 1 \otimes x_k) \\ &= \bar{f}_k \bar{f}_\ell - \bar{f}_\ell \bar{f}_k \end{aligned}$$

since maps of the form $? \otimes 1$ and $1 \otimes ?$, respectively, commute; this shows that $\bar{H}_{\{k<\ell\}}$ is a homotopy from $\bar{f}_\ell \bar{f}_k$ to $\bar{f}_k \bar{f}_\ell$ as required (we have of course used that $H_{\{k<\ell\}}$ is a homotopy from $f_\ell f_k$ to $f_k f_\ell$ as F is a special N -cube.)

For $S \subset N$ with $s \geq 3$ we observe that

$$\begin{aligned}
& (-1)^s [S : \emptyset] \cdot \bar{d} \circ \bar{H}_S + [S : \emptyset] \cdot \bar{H}_S \circ \bar{d} \\
& + \sum_{z \in S} [S : S \setminus z] [S \setminus z : \emptyset] \cdot \bar{f}_z \circ \bar{H}_{S \setminus z} \\
& + (-1)^s \sum_{z \in S} [S : z] \cdot \bar{H}_{S \setminus z} \circ \bar{f}_z \\
& + \sum_{\substack{T \subseteq S \\ \bullet \\ t \geq 2 \leq s-t}} (-1)^{ts} [S : T] [T : \emptyset] \cdot \bar{H}_{S \setminus T} \circ \bar{H}_T
\end{aligned}$$

is the same as

$$\begin{aligned}
& \left((-1)^s [S : \emptyset] \cdot d \circ H_S + [S : \emptyset] \cdot H_S \circ d \right. \\
& + \sum_{z \in S} [S : S \setminus z] [S \setminus z : \emptyset] \cdot f_z \circ H_{S \setminus z} \\
& + (-1)^s \sum_{z \in S} [S : z] \cdot H_{S \setminus z} \circ f_z \\
& + \sum_{\substack{T \subseteq S \\ \bullet \\ t \geq 2 \leq s-t}} (-1)^{ts} [S : T] [T : \emptyset] \cdot H_{S \setminus T} \circ H_T \Big) \otimes 1 \\
& - \left(\sum_{z \in S} ([S : S \setminus z] [S \setminus z : \emptyset] + (-1)^s [S : z]) \cdot H_{S \setminus z} \otimes x_z \right);
\end{aligned}$$

to which the first four lines contribute nothing as F is a special N -cube, using Corollary I.4.3. So it is enough to verify that

$$[S : S \setminus z] [S \setminus z : \emptyset] + (-1)^s [S : z] = 0$$

for every $z \in S$, which is a pleasant combinatorial exercise left to the interested reader. \square

II.4. THE MATHER TRICK FOR MAPPING TORI

Let $h_1, \dots, h_n: D \longrightarrow D$ be mutually commuting self-maps of the R -module cochain complex D . We denote the mapping torus of the trivial special N -cube $\mathfrak{T}\mathfrak{r}\mathfrak{i}\mathfrak{v}(D; h_1, h_2, \dots, h_n)$ by X , and denote the differential of X by $D^X = (D_{B,A}^X)_{A,B \subseteq N}$.

Let $g: D \longrightarrow D$ be another cochain map, and let $G: \text{id} \simeq g$ be a homotopy so that $dG + Gd = g - \text{id}$. We can now form the derived special N -cube $\mathfrak{D}\mathfrak{e}\mathfrak{r}(D; g, \text{id}_D, G; h_1, h_2, \dots, h_n)$ according to Theorem I.5.1 and Definition I.5.2, with $\beta = \text{id}_D$ and $\alpha = g$. We denote its mapping torus by Y , which has differential $D^Y = (D_{B,A}^Y)_{A,B \subseteq N}$. The complex Y is defined in terms of certain maps \bar{d} , \bar{f}_k and \bar{H}_S as prescribed in Definition II.3.1.

Finally, we define a matrix $K = (K_{B,A})_{A,B \subseteq N}$ which, *a priori*, is just a collection of module homomorphisms $Y^n \longrightarrow X^n$:

$$K_{B,A} = M_{B,A} \otimes 1$$

where $M_{B,A}$ is as defined in (II.2.1) and (II.2.2).

Lemma II.4.1. *The matrix K defines a cochain map*

$$\begin{aligned} \mathcal{T} \mathfrak{Dct}(D; g, \text{id}_D, G; h_1, h_2, \dots, h_n) &= Y \\ &\xrightarrow{K} X = \mathcal{T} \mathfrak{Triv}(D; h_1, h_2, \dots, h_n) . \end{aligned}$$

This map is a quasi-isomorphism. If D is a bounded above complex of projective R -modules, then K is a homotopy equivalence $K: Y \simeq X$.

Proof. Let us remark first that the present Lemma is not a special case of Lemma II.2.3: the procedure involved in the construction of mapping tori is not compatible with taking derived cubes in any obvious way. Nevertheless, the statements are quite similar, and we concentrate on giving the necessary modifications in the proof only. The main point is to check that K defines a cochain map; to this end we need to verify that

$$(KD^Y)_{B,A} - (D^X K)_{B,A} = 0$$

for all $A \subseteq B \subseteq N$.

We focus on $(D^X K)_{B,A} = \sum_{B \supseteq S \supseteq A} D_{B,S}^X K_{S,A}$ first, and assume that $b - a \geq 3$. As X is the totalisation of a trivial N -cube, cf. Example II.3.2, we know that $D_{B,S}^X = 0$ unless $\#(B \setminus S) \leq 1$, and in case of equality $D_{B,S}^X$ is of the form $\pm(h_z \otimes 1 - 1 \otimes x_z)$. All this means that $(D^X K)_{B,A}$ equals

$$\begin{aligned} &((-1)^b d \cdot (-1)^b (-1)^{ab} [B : A] \cdot G \circ H_{B \setminus A}) \otimes 1 \\ &+ \sum_{z \in B \setminus A} ([B : B \setminus z] \cdot h_z \circ (-1)^{b-1} (-1)^{a(b-1)} [B \setminus z : A] \cdot G \circ H_{(B \setminus z) \setminus A}) \otimes 1 \\ &- \sum_{z \in B \setminus A} ([B : B \setminus z] \cdot (-1)^{b-1} (-1)^{a(b-1)} [B \setminus z : A] \cdot G \circ H_{(B \setminus z) \setminus A}) \otimes x_z \end{aligned}$$

so that, simplifying the sign terms and introducing a term of the form $GdH_{B \setminus A} - GdH_{B \setminus A}$, we obtain

$$\begin{aligned} (-1)^{ab} (D^X K)_{B,A} &= \left([B : A] ((dG + Gd) \circ H_{B \setminus A} - GdH_{B \setminus A}) \right) \otimes 1 \\ &- (-1)^{b-a} \cdot \sum_{z \in B \setminus A} ([B : B \setminus z] [B \setminus z : A] \cdot h_z \circ G \circ H_{(B \setminus z) \setminus A}) \otimes 1 \\ &+ (-1)^{b-a} \cdot \sum_{z \in B \setminus A} ([B : B \setminus z] [B \setminus z : A] \cdot G \circ H_{(B \setminus z) \setminus A}) \otimes x_z . \end{aligned}$$

The second line of this expression reduces to $([B : A] \cdot H_{B \setminus A}) \otimes 1$, as we showed in the proof of Lemma II.2.3; as $dG + Gd = g - \text{id}$ this means that

$$\begin{aligned} (D^X K)_{B,A} &= (-1)^{ab} [B : A] \cdot (gH_{B \setminus A} - GdH_{B \setminus A}) \otimes 1 \\ &+ (-1)^{ab} (-1)^{b-a} \cdot \sum_{z \in B \setminus A} ([B : B \setminus z] [B \setminus z : A] \cdot G \circ H_{(B \setminus z) \setminus A}) \otimes x_z . \end{aligned}$$

Plugging in all relevant definitions, and the expression for $(D^X K)_{B,A}$ we just obtained, we see that the difference $(KD^Y)_{B,A} - (D^X K)_{B,A}$ is given by

a sum of two expressions: the tensor product of the sum (II.2.5) with the identity map of $L = R[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$, and (up to a factor of $-(-1)^{ab}$)

$$\sum_{z \in B \setminus A} ((-1)^{b-a} [B : B \setminus z] [B \setminus z : A] + [B : A \amalg z] [A \amalg z : A]) \cdot (G \circ H_{(B \setminus z) \setminus A}) \otimes x_z.$$

We know from the proof of Lemma II.2.3 that the former is trivial; to show that the latter is trivial as well it is enough to verify the equality

$$[B : B \setminus z] [B \setminus z : A] + (-1)^{b-a} [B : A \amalg z] [A \amalg z : A] = 0. \quad (\text{II.4.2})$$

The left-hand side acquires a sign independent of z when removing an element of A from both A and B , by Lemma I.1.4, so that we may assume without loss of generality that $A = \emptyset$ and $a = 0$. But as $[z : \emptyset] = 1$ this then is precisely the pleasant combinatorial exercise the reader solved at the end of the proof of Lemma II.3.4.

The verification for $b-a < 3$ can be done along similar lines, but is slightly easier due to the lower number of terms involved. We omit the details. \square

We introduce yet more notation: suppose we are given, in addition to the maps h_i above, cochain maps $\alpha : C \longrightarrow D$ and $\beta : D \longrightarrow C$. We set $g = \alpha \circ \beta : D \longrightarrow D$, and let G as before be a homotopy $G : \text{id}_D \simeq \alpha \circ \beta = g$ so that $dG + Gd = \alpha\beta - \text{id}$. This gives us the data required to define the special N -cube $\mathfrak{Dcr}(C; \alpha, \beta, G; h_1, h_2, \dots, h_n)$. We denote its mapping torus by Z , and the corresponding differential by $D^Z = (D_{B,A}^Z)_{A,B \subseteq N}$.

We let J denote the (constant) diagonal matrix with entry $\beta \otimes 1$; this is to be considered as a module homomorphism $Y^n \longrightarrow Z^n$.

Lemma II.4.3 (MATHER trick for mapping tori). *The matrix J defines a cochain map*

$$\begin{aligned} \mathcal{T} \mathfrak{Dcr}(D; \alpha\beta, \text{id}_D, G; h_1, h_2, \dots, h_n) &= Y \\ &\xrightarrow{J} Z = \mathcal{T} \mathfrak{Dcr}(C; \alpha, \beta, G; h_1, h_2, \dots, h_n). \end{aligned}$$

If β is a quasi-isomorphism so is J . If in addition C and D are bounded above complexes of projective R -modules, then J is a homotopy equivalence of L -module complexes $J : Y \simeq Z$.

Proof. This is straightforward; note that maps of the form $1 \otimes ?$ and $? \otimes 1$ commute. \square

Part III. Novikov homology and finite domination

Before attacking the main topic of the paper, the relationship between finite domination and NOVIKOV cohomology, we need to digress and introduce a few auxiliary algebraic constructions: truncated products, multi-complexes, and totalisation of multi-complexes.

III.1. TRUNCATED PRODUCTS

The set \mathcal{P} of formal LAURENT series $f = \sum_{\mathbf{a} \in \mathbb{Z}^n} r_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$, where $r_{\mathbf{a}} \in R$ and

$$\mathbf{x}^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \dots x_n^{a_n} \text{ for } \mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{Z}^n,$$

has an obvious module structure over the LAURENT polynomial ring $L = R[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$ given by multiplication and collecting terms. We define the *support* $\text{supp}(f)$ of a formal LAURENT series f to be the set of those $\mathbf{a} \in \mathbb{Z}^n$ with $r_{\mathbf{a}} \neq 0$. The NOVIKOV ring $R((x_1, x_2, \dots, x_n))$ is defined as

$$\begin{aligned} R((x_1, x_2, \dots, x_n)) &= \{f \in \mathcal{P} \mid \exists k \in \mathbb{N}: k\mathbf{1} + \text{supp}(f) \subseteq \mathbb{N}^n\} \\ &= R[[x_1, x_2, \dots, x_n]][1/(x_1 x_2 \cdots x_n)] ; \end{aligned}$$

here we write $\mathbf{1} = (1, 1, \dots, 1) \in \mathbb{N}^n$ and $v + A = \{v + a \mid a \in A\}$.

Given a \mathbb{Z}^n -indexed family of R -modules $M_{\mathbf{a}}$, we write the elements of the infinite product $\prod_{\mathbf{a} \in \mathbb{Z}^n} M_{\mathbf{a}}$ as formal LAURENT series $g = \sum_{\mathbf{a} \in \mathbb{Z}^n} m_{\mathbf{a}} x^{\mathbf{a}}$, where $m_{\mathbf{a}} \in M_{\mathbf{a}}$ corresponds to the factor indexed by $\mathbf{a} \in \mathbb{Z}^n$. The support of g is defined as above. The *truncated product* is defined as

$$\prod_{\text{tr}}_{\mathbf{a} \in \mathbb{Z}^n} M_{\mathbf{a}} = \{g \in \prod_{\mathbf{a} \in \mathbb{Z}^n} M_{\mathbf{a}} \mid \exists k \in \mathbb{N}: k\mathbf{1} + \text{supp}(g) \subseteq \mathbb{N}^n\} .$$

For later use we formulate this in slightly more fancy terms. Given a vector $(a_1, a_2, \dots, a_n) = \mathbf{a} \in \mathbb{Z}^n$ let us introduce the symbol

$$[\mathbf{a}] = \min_{1 \leq i \leq n} a_i . \quad (\text{III.1.1})$$

Given an element $g \in \prod M_{\mathbf{a}}$ and an integer $k \in \mathbb{Z}$ we write $[\text{supp}(g)] \geq k$ if every $\mathbf{a} \in \text{supp}(g)$ satisfies $[\mathbf{a}] \geq k$; we say $[\text{supp}(g)] = k$ if in addition $[\text{supp}(g)] \not\geq k + 1$. — With this notation $\prod_{\text{tr}}_{\mathbf{a} \in \mathbb{Z}^n} M_{\mathbf{a}}$ consists of those elements $g \in \prod M_{\mathbf{a}}$ such that there exists $k \in \mathbb{Z}$, depending on g , with $[\text{supp}(g)] \geq k$.

Of particular interest is the case of a *truncated power*, having $M_{\mathbf{a}} = M$ for all $\mathbf{a} \in \mathbb{Z}^n$; we reserve the notation

$$M((x_1, x_2, \dots, x_n)) = \prod_{\text{tr}}_{\mathbb{Z}^n} M$$

for this. The truncated power $M((x_1, x_2, \dots, x_n))$ comes equipped with a natural $R((x_1, x_2, \dots, x_n))$ -module structure described by multiplication of formal LAURENT series and using the scalar action of R on M . In the case $M = R$ we obtain an equality of the NOVIKOV ring $R((x_1, x_2, \dots, x_n))$ with the truncated power $\prod_{\text{tr}}_{\mathbb{Z}^n} R$.

Lemma III.1.2. *For a finitely presented R -module M there is a canonical isomorphism of $R((x_1, x_2, \dots, x_n))$ -modules*

$$M \otimes_R R((x_1, x_2, \dots, x_n)) \cong M((x_1, x_2, \dots, x_n)) ,$$

sending the elementary tensor $m \otimes \sum_{\mathbf{a} \in \mathbb{Z}^n} r_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$ to the formal LAURENT series $\sum_{\mathbf{a} \in \mathbb{Z}^n} (m \cdot r_{\mathbf{a}}) \mathbf{x}^{\mathbf{a}} \in M((x_1, x_2, \dots, x_n))$.

Proof. The proof is standard; details for the case $n = 1$ have been recorded, for example, in [Hüt11, Lemma 2.1]. One establishes the result for finitely generated free R -modules first, and then passes to the general case by considering a two-step resolution of M by finitely generated free modules. \square

III.2. MULTI-COMPLEXES AND THEIR TOTALISATION

An $(n + 1)$ -complex is a family of R -modules $E^\bullet = (E^{\mathbf{b}})$, indexed by $(n + 1)$ -tuples $\mathbf{b} = (b_1, b_2, \dots, b_{n+1}) \in \mathbb{Z}^{n+1}$, together with R -module homomorphisms

$$d_i: E^{\mathbf{b}} \longrightarrow E^{\mathbf{b} + \mathbf{e}_i}, \quad 1 \leq i \leq n + 1,$$

where $\mathbf{e}_i = (0, \dots, 0, 1, 0, \dots, 0)$ is the i th unit vector, satisfying

$$d_i \circ d_i = 0, \quad \text{and} \quad d_i d_j = -d_j d_i \text{ for } 1 \leq i < j \leq n + 1.$$

Its *direct sum totalisation* is the cochain complex $\text{Tot } E^\bullet$ defined by

$$(\text{Tot } E^\bullet)^k = \bigoplus_{a_1, \dots, a_n \in \mathbb{Z}} E^{[\mathbf{a}, k]},$$

where $[\mathbf{a}, k] = (a_1, a_2, \dots, a_n, k - a_1 - a_2 - \dots - a_n)$, with differential $d = d_1 + d_2 + \dots + d_{n+1}$.

The totalisation of a commutative N -diagram can be described as the totalisation of an $(n + 1)$ -complex; this is a matter of checking sign conventions with the help of Lemma I.1.3. We record this fact:

Proposition III.2.1. *Let C be a complex of R -modules, and let f_i , $1 \leq i \leq n$, be a collection of pairwise commuting cochain self-maps of C . The totalisation of $\mathfrak{T}\mathfrak{riv}(C; f_1, f_2, \dots, f_n)$ is the totalisation of an $(n + 1)$ -fold cochain complex E^\bullet which has the module C^k in degrees $(\epsilon_1, \epsilon_2, \dots, \epsilon_n, k)$ where $\epsilon_i \in \{0, 1\}$ and $k \in \mathbb{Z}$. The differential in \mathbf{e}_{n+1} -direction is given by the differential of C modified by the sign $(-1)^{\epsilon_1 + \epsilon_2 + \dots + \epsilon_n}$. The differential d_k for $1 \leq k \leq n$ is given by $d_k = (-1)^{\epsilon_1 + \epsilon_2 + \dots + \epsilon_{k-1}} \cdot f_k$. \square*

Corollary III.2.2. *If one of the maps f_i is a quasi-isomorphism, or if C is acyclic, then $\text{Tot } \mathfrak{T}\mathfrak{riv}(C; f_1, f_2, \dots, f_n)$ is acyclic.*

Proof. This is standard homological algebra of multi-complexes. One way to prove the claim is to use a “partial totalisation”. Let E^\bullet be the multi-complex described in Proposition III.2.1. If, for example, f_1 is a quasi-isomorphism, we may define a 2-complex $D^{*,*}$ by setting

$$D^{p,q} = \bigoplus_{a_2, a_3, \dots, a_n \in \mathbb{Z}} E^{p, a_2, a_3, \dots, a_n, q - \sum_i a_i},$$

equipped with horizontal differential d_h given by a direct sum of maps d_1 and vertical differential d_v induced by $d_2 + d_3 + \dots + d_{n+1}$. This 2-complex is concentrated in the two columns $p = 0$ and $p = 1$, and has acyclic rows by our hypothesis on f_1 . Hence its totalisation, which is the same as the totalisation of E^\bullet , is acyclic. \square

The technique used to prove the Corollary for the case of acyclic C is of interest later on. We record a variant for later use:

Corollary III.2.3. *Let C and f_i be as in Proposition III.2.1. Then the 2-complex*

$$D^{p,q} = \bigoplus_{\substack{A \subseteq N \\ \#A=p}} C^q$$

with vertical differential

$$d^v = \bigoplus_{\substack{A \subseteq N \\ \#A=p}} (-1)^p d_C : D^{p,q} \longrightarrow D^{p,q+1}$$

and horizontal differential given by $[A \amalg j : A] \cdot f_j$ (for $j \notin A$) when considered as a map from A -summand to $(A \amalg j)$ -summand, satisfies

$$\mathrm{Tot} D^\bullet = \mathrm{Tot} \mathfrak{I} \mathrm{riv}(C; f_1, f_2, \dots, f_n) .$$

This 2-complex D^\bullet is concentrated in columns $0 \leq p \leq n$. □

Back to a general $(n+1)$ -complex E^\bullet , its *truncated product totalisation* ${}_{\mathrm{tr}} \mathrm{Tot} E^\bullet$ is defined similar to Tot , using truncated products in place of direct sums:

$$({}_{\mathrm{tr}} \mathrm{Tot} E^\bullet)^k = \prod_{\mathbf{a} \in \mathbb{Z}^n} E^{[\mathbf{a}, k]} ,$$

where $[\mathbf{a}, k] = (a_1, a_2, \dots, a_n, k - a_1 - a_2 - \dots - a_n) \in \mathbb{Z}^{n+1}$, and with differential again given by $d = d_1 + d_2 + \dots + d_{n+1}$.

Proposition III.2.4. *Suppose that the $(n+1)$ -complex E^\bullet is such that for any $a_1, a_2, \dots, a_n \in \mathbb{Z}$ the cochain complex $E^{a_1, a_2, \dots, a_n, *}$ with differential d_{n+1} is exact (that is, E^\bullet is exact in \mathbf{e}_{n+1} -direction). Then its truncated product totalisation ${}_{\mathrm{tr}} \mathrm{Tot} E^\bullet$ is acyclic.*

Proof. Let $c \in ({}_{\mathrm{tr}} \mathrm{Tot} E^\bullet)^m = \prod_{\mathbf{a} \in \mathbb{Z}^n} E^{[\mathbf{a}, m]}$ be a cocycle so that $d(c) = 0$. We want to construct an element $b = (b_{\mathbf{a}})_{\mathbf{a} \in \mathbb{Z}^n} \in ({}_{\mathrm{tr}} \mathrm{Tot} E^\bullet)^{m-1}$ with $d(b) = c$. — We have $[\mathrm{supp}(c)] = k$, for some $k \in \mathbb{Z}$; we set $b_{\mathbf{a}} = 0 \in E^{[\mathbf{a}, m-1]}$ whenever $[\mathbf{a}] < k$. These elements trivially satisfy $d(b)_{\mathbf{a}} = c_{\mathbf{a}}$ for $[\mathbf{a}] < k$. (Note here that while not all components of b have been defined yet, the expression

$$d(b)_{\mathbf{a}} = d_{n+1}(b_{\mathbf{a}}) + \sum_{i=1}^n d_i(b_{\mathbf{a} - \mathbf{e}_i})$$

makes sense as $[\mathbf{a} - \mathbf{e}_i] \leq [\mathbf{a}] < k$.)

We need to find suitable elements $b_{\mathbf{a}} \in E^{[\mathbf{a}, m-1]}$ for $[\mathbf{a}] \geq k$. We proceed by induction on $\ell = |\mathbf{a}| \geq nk$, where $|\mathbf{a}| = \sum_1^n a_i$.

So suppose $[\mathbf{a}] \geq k$. For $\ell = nk$ we necessarily have $\mathbf{a} = (k, k, \dots, k)$ and consequently $[\mathbf{a} - \mathbf{e}_j] < k$. We calculate, using the definition of d and

the equalities $d_j \circ d_j = 0$ and $d_i \circ d_j = -d_j \circ d_i$ for $i \neq j$,

$$\begin{aligned}
0 &= d(c)_{\mathbf{a}} \\
&= d_{n+1}(c_{\mathbf{a}}) + \sum_{j=1}^n d_j(c_{\mathbf{a}-\mathbf{e}_j}) \\
&= d_{n+1}(c_{\mathbf{a}}) + \sum_{j=1}^n d_j(d(b)_{\mathbf{a}-\mathbf{e}_j}) \\
&= d_{n+1}(c_{\mathbf{a}}) + \sum_{j=1}^n d_j \left(d_{n+1}(b_{\mathbf{a}-\mathbf{e}_j}) + \sum_{i=1}^n d_i(b_{\mathbf{a}-\mathbf{e}_j-\mathbf{e}_i}) \right) \quad (\text{III.2.5}) \\
&= d_{n+1}(c_{\mathbf{a}}) + \sum_{j=1}^n d_j d_{n+1}(b_{\mathbf{a}-\mathbf{e}_j}) \\
&= d_{n+1} \left(c_{\mathbf{a}} - \sum_{j=1}^n d_j(b_{\mathbf{a}-\mathbf{e}_j}) \right)
\end{aligned}$$

so that, by our exactness hypothesis, we find an element $b_{\mathbf{a}} \in E^{[\mathbf{a}, m-1]}$ with

$$d_{n+1}(b_{\mathbf{a}}) = c_{\mathbf{a}} - \sum_{j=1}^n d_j(b_{\mathbf{a}-\mathbf{e}_j}) .$$

Then by definition of d and the construction of $b_{\mathbf{a}}$ we have

$$d(b)_{\mathbf{a}} = d_{n+1}(b_{\mathbf{a}}) + \sum_{j=1}^n d_j(b_{\mathbf{a}-\mathbf{e}_j}) = c_{\mathbf{a}} .$$

Assume now, by induction, that for some $\ell > nk$ we have already constructed elements $b_{\mathbf{b}}$ for $|\mathbf{b}| \geq k$ and $|\mathbf{b}| < \ell$ satisfying $d(b)_{\mathbf{b}} = c_{\mathbf{b}}$. Then for any $\mathbf{a} \in \mathbb{Z}^n$ with $|\mathbf{a}| \geq k$ and $|\mathbf{a}| = \ell$ we notice that either $|\mathbf{a} - \mathbf{e}_j| < k$ or else $|\mathbf{a} - \mathbf{e}_j| < \ell$. So using our induction hypothesis, the calculation (III.2.5) remains valid for our current \mathbf{a} , which allows us to find the requisite element $b_{\mathbf{a}}$ in exactly the same manner as before.

We have now defined elements $b_{\mathbf{a}} \in E^{[\mathbf{a}, m-1]}$ for all $\mathbf{a} \in \mathbb{Z}^n$, satisfying $d(b) = c$ by construction. As c was arbitrary, this proves that any cocycle of $\text{tr Tot}(E^\bullet)$ is a coboundary. \square

III.3. FROM MAPPING TORI TO MULTI-COMPLEXES

In this paper multi-complexes are mainly used as a tool to get alternative representations of mapping tori. To set the stage for the following construction, let us introduce a LAURENT polynomial notation for the elements of \mathbb{Z}^n -indexed copowers of an R -module M much in the spirit of §III.1:

$$\begin{aligned}
M[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}] &= \bigoplus_{\mathbf{a} \in \mathbb{Z}^n} M \\
&= \left\{ \sum_{\mathbf{a} \in \mathbb{Z}^n} m_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} \mid m_{\mathbf{a}} = 0 \text{ for almost all } \mathbf{a} \right\}
\end{aligned}$$

The element $m_{\mathbf{a}}$ belongs of course to the summand indexed by \mathbf{a} . We have an obvious isomorphism

$$M \otimes_R R[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}] \cong M[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}], \quad (\text{III.3.1})$$

sending the elementary tensor $m \otimes \sum_{\mathbf{a} \in \mathbb{Z}^n} r_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$ to the formal LAURENT polynomial $\sum_{\mathbf{a} \in \mathbb{Z}^n} (m \cdot r_{\mathbf{a}}) \mathbf{x}^{\mathbf{a}} \in M[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$.

Now let F be a special N -cube on the R -module cochain complex C , specified by the usual data of differential $d = d_C$, cochain maps f_i , and (higher) homotopies H_S . Let $T = \text{Tot } F$ denote the totalisation of F . Define an $(n+1)$ -multi-complex $\mathbf{L}(F)^{\bullet}$ by saying that *at position $\mathbf{a} \in \mathbb{Z}^n$, the complex $\mathbf{L}(F)^{\mathbf{a},*}$ in \mathbf{e}_{n+1} -direction is the shift $T[|\mathbf{a}|]$ of T , where $T[|\mathbf{a}|]^{\ell} = T^{\ell+|\mathbf{a}|}$, with differential as in T re-indexed suitably. We need to specify the differential in \mathbf{e}_k -direction, $1 \leq k \leq n$.⁷ Recall that, by definition of totalisation of N -cubes, we have*

$$\mathbf{L}(F)^{\mathbf{a},\ell} = T^{\ell+|\mathbf{a}|} = \bigoplus_{A \subseteq N} C^{\ell+|\mathbf{a}|-a}$$

and similarly

$$\mathbf{L}(F)^{\mathbf{a}+\mathbf{e}_k,\ell} = T^{\ell+|\mathbf{a}|+1} = \bigoplus_{A \subseteq N} C^{\ell+|\mathbf{a}|+1-a}.$$

Now the differential d_k in \mathbf{e}_k -direction restricted to the A -summand is trivial if $k \in A$. Otherwise, it is given by multiplication with the sign $-[A \amalg k : A]$ followed by inclusion into the $A \amalg \{k\}$ -summand.

The differentials anti-commute. To show that, it is enough to compare $d_k d_{\ell}$ and $d_{\ell} d_k$ considered as maps from A -summand to B -summand, for $A, B \subseteq N$ (that is, we restrict to the A -summand and co-restrict to the B -summand). For $k, \ell \leq n$ both composites are zero, by definition of the differentials, unless $B = A \amalg \{k, \ell\}$, in which case $d_k d_{\ell} = -d_{\ell} d_k$ by the simplicial identities (I.1.2). For $\ell = n+1$ and $k \leq n$ we need to recall the definition of the differential in the totalisation of a special N -cube as the matrix $\left((-1)^{st} [T : S] H_{T \setminus S} \right)_{N \supseteq T \supseteq S}$. By definition of d_k we only need to consider the case $B \ni k \notin A$. The composition $d_{n+1} d_k$ gives us

$$(-1)^{b(a+1)} [B : A \amalg k] H_{B \setminus (A \amalg k)} \cdot (-[A \amalg k : A])$$

while we have

$$(-[B : B \setminus k]) \cdot (-1)^{a(b-1)} [B \setminus k : A] H_{(B \setminus k) \setminus A}$$

for the composition $d_k d_{n+1}$. To say that these have opposite sign amounts to saying that

$$[B : B \setminus k] [B \setminus k : A] + (-1)^{b-a} [B : A \amalg k] [A \amalg k : A] = 0,$$

after cancelling a common factor of $(-1)^{ab}$. Now for $b-a=1$ (so that $B = A \amalg k$) this is trivial, for $b-a=2$ this is a reformulation of the simplicial identities (I.1.2), and for $b-a \geq 3$ this is identity (II.4.2) again.

Proposition III.3.2. *The isomorphism (III.3.1) induces an isomorphism of $R[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$ -module complexes $\text{Tot } \mathbf{L}(F)^{\bullet} \cong \mathcal{T}F$.*

⁷We are using $N = \{1, 2, \dots, n\}$ here

Proof. By construction we have

$$(\mathrm{Tot} \mathbf{L}(F)^\bullet)^k = \bigoplus_{a_1, \dots, a_n \in \mathbb{Z}} \mathbf{L}(F)^{[\mathbf{a}, k]},$$

where $[\mathbf{a}, k] = (a_1, a_2, \dots, a_n, k - a_1 - a_2 - \dots - a_n)$; but $\mathbf{L}(F)^{[\mathbf{a}, k]} = T^k$, by definition of $\mathbf{L}(F)^\bullet$, so that

$$(\mathrm{Tot} \mathbf{L}(F)^\bullet)^k = \bigoplus_{\mathbf{a} \in \mathbb{Z}^n} T^k = T^k[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}] \stackrel{(\text{III.3.1})}{\cong} T^k \otimes_R L.$$

Next we note, using distributivity of tensor products, that $T \otimes_R L$ is the cochain complex underlying the special N -cube used to define the mapping torus of F :

$$\begin{aligned} (T \otimes_R L)^k &= T^k \otimes_R L = \left(\bigoplus_{A \subseteq N} C^{k-a} \right) \otimes_R L \\ &\cong \bigoplus_{A \subseteq N} (C^{k-a} \otimes_R L) = \bigoplus_{A \subseteq N} (C \otimes_R L)^{k-a} \end{aligned}$$

It is now a matter of tedious but straightforward checking that under these identifications the differentials $d_1 + d_2 + \dots + d_{n+1}$ of $\mathrm{Tot} \mathbf{L}(F)^\bullet$ and of $\mathcal{T}F = \mathrm{Tot} \bar{F}$, with \bar{F} as in Definition II.3.1, agree. Indeed, the action of the maps $1 \otimes x_k$ is encoded in the differential d_k , while the effect of all other structure maps of F (the differential $d \otimes 1$, the maps $f_k \otimes 1$, the homotopies $H_S \otimes 1$) is captured by the differential in \mathbf{e}_{n+1} -direction. \square

III.4. REPLACING L -MODULE COMPLEXES BY MAPPING TORI

We want to show that any cochain complex D of modules over the LAURENT polynomial ring $L = R[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$ can be written as a mapping torus. We can, by restriction of scalars, consider D as a complex of R -modules, as will often be done in the sequel. In particular, we can form a new L -module cochain complex $D \otimes_R L$. Note that the n self-maps $x_k \otimes \mathrm{id} - \mathrm{id} \otimes x_k : D \otimes_R L \longrightarrow D \otimes_R L$ commute pairwise.

The mapping torus $\mathcal{T} \mathfrak{Triv}(D; x_1, x_2, \dots, x_n)$ is, by definition, the totalisation of the commutative N -cubical diagram

$$\mathfrak{Triv}(D \otimes_R L; x_1 \otimes \mathrm{id} - \mathrm{id} \otimes x_1, x_2 \otimes \mathrm{id} - \mathrm{id} \otimes x_2, \dots, x_n \otimes \mathrm{id} - \mathrm{id} \otimes x_n),$$

with module in cochain level m being given by $\bigoplus_{A \subseteq N} D^{m-a} \otimes_R L$. The assignment $z \otimes p \mapsto z \cdot p$ on the N -summand, and $z \otimes p \mapsto 0$ on all other summands, defines an L -linear cochain map

$$\psi : \mathcal{T} \mathfrak{Triv}(D; x_1, x_2, \dots, x_n) \longrightarrow \Sigma^n D.$$

Lemma III.4.1. *The map ψ is a quasi-isomorphism. It is a homotopy equivalence if D is a bounded above complex of projective L -modules.*

Proof. Re-write the mapping torus $\mathcal{T} \mathfrak{Triv}(D; x_1, x_2, \dots, x_n)$ as the totalisation of the 2-complex D^\bullet associated to the commutative diagram

$$\mathfrak{Triv}(D \otimes_R L; x_1 \otimes \mathrm{id} - \mathrm{id} \otimes x_1, x_2 \otimes \mathrm{id} - \mathrm{id} \otimes x_2, \dots, x_n \otimes \mathrm{id} - \mathrm{id} \otimes x_n)$$

according to Corollary III.2.3. Let \hat{D}^\bullet denote the 2-complex which agrees with D^\bullet everywhere except for the $(n+1)$ st column where $\hat{D}^{n+1,q} = D^q$; the new differentials are given as follows:

$$\begin{aligned} d_h: \hat{D}^{n,q} = D^q \otimes_R L &\xrightarrow{\psi} D^q = \hat{D}^{n+1,q} \\ d_v: \hat{D}^{n+1,q} = D^q &\xrightarrow{(-1)^{n+1}d_D} D^{q+1} = \hat{D}^{n+1,q+1} \end{aligned}$$

Then $\text{Tet } \hat{D}^\bullet$ is isomorphic to the mapping cone of ψ , whence it is enough to show that $\text{Tet } \hat{D}^\bullet$ is acyclic. For this it is clearly sufficient to verify that \hat{D}^\bullet has exact rows; we may in fact, without loss of generality, restrict attention to the 0th row $q = 0$.

So the claim to verify is the following: *Given an L -module $M = D^0$, the map*

$$\psi_M = \psi: D^{*,0} \longrightarrow M[n], \quad D^{n,0} = M \otimes_R L \ni z \otimes p \mapsto z \cdot p$$

is a quasi-isomorphism from the 0th row of D^\bullet to the module M , considered as a cochain complex concentrated in degree n .

The crucial observation is that the 0th row $D^{*,0}$ of D^\bullet is nothing but the mapping torus $\mathcal{T} \mathfrak{Triv}(M; x_1, x_2, \dots, x_n)$ of a trivial N -cube. We make use of this fact as follow. Let $S = R[x_1, x_2, \dots, x_n]$ be the polynomial ring, and let F denote the commutative cubical diagram $\mathfrak{Triv}(S; x_1, x_2, \dots, x_n)$, considered as a special N -cube of R -module complexes. Let $L(F)^\bullet$ be the multi-complex associated to F according to §III.3. Then, as we have seen in Proposition III.3.2, $\mathcal{T}(F) \cong \text{Tet } L(F)^\bullet$. By construction $L(F)^\bullet$ has a shifted copy of the cochain complex $K = \text{Tet } (F)$ in $(n+1)$ -direction everywhere. Now K is actually the KOSZUL complex of S associated with the regular sequence (x_1, x_2, \dots, x_n) so that K is quasi-isomorphic, via the canonical projection, to $S/(x_1, x_2, \dots, x_n)[n] = R[n]$ (that is, the module R considered as a complex concentrated in degree n); since K consists of free R -modules, this quasi-isomorphism is actually a homotopy equivalence of R -module complexes. Upon taking \mathbb{Z}^n -indexed copowers we thus obtain a homotopy equivalence

$$\epsilon: \mathcal{T}(F) \xrightarrow{\simeq} R[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}][n] = L[n]$$

of L -module complexes. Note that the cochain modules of $\mathcal{T}(F)$ are direct sums of modules of the type $S \otimes_R L$. Thus we can form the map $M \otimes_S \epsilon$, tensoring source and target of ϵ over S with M . The source of $M \otimes_S \epsilon$ is canonically isomorphic to the mapping torus of $\mathfrak{Triv}(M; x_1, x_2, \dots, x_n)$, the target $M \otimes_S L[n]$ is canonically isomorphic to $M[n]$, and $\psi_M = M \otimes \epsilon$ is a homotopy equivalence of L -module complexes as required. \square

Corollary III.4.2. *Suppose that we are given a complex D of L -modules, an R -module complex C , and mutually inverse homotopy equivalences of R -module complexes*

$$\alpha: C \longrightarrow D \quad \text{and} \quad \beta: D \longrightarrow C.$$

Let $G: \text{id}_D \simeq \alpha \circ \beta$ be a homotopy from id_D to $\alpha \circ \beta$. Then the cochain complex $\Sigma^n D$ is quasi-isomorphic by L -linear maps to the complex

$$\mathcal{T} \mathfrak{Det}(C; \alpha, \beta, G; x_1, x_2, \dots, x_n).$$

If in addition both C and D are bounded above and consist of projective R -modules (restricting the L -module structure to R in the case of D) the complexes $\Sigma^n D$ and $\mathcal{T}\mathfrak{D}\mathfrak{ct}(C; \alpha, \beta, G; x_1, x_2, \dots, x_n)$ are actually homotopy equivalent as L -module complexes.

Proof. We have a chain of quasi-isomorphisms

$$\begin{aligned} \Sigma^n D &\xleftarrow[\text{III.4.1}]{\sim} \mathcal{T}\mathfrak{I}\mathfrak{v}(D; x_1, x_2, \dots, x_n) \\ &\xleftarrow[\text{II.4.1}]{\sim} \mathcal{T}\mathfrak{D}\mathfrak{ct}(D; \alpha\beta, \text{id}, G; x_1, x_2, \dots, x_n) \\ &\xrightarrow[\text{II.4.3}]{\sim} \mathcal{T}\mathfrak{D}\mathfrak{ct}(C; \alpha, \beta, G; x_1, x_2, \dots, x_n) . \end{aligned}$$

The final part is automatic. \square

III.5. FINITE DOMINATION IMPLIES VANISHING OF NOVIKOV COHOMOLOGY

Theorem III.5.1. *Suppose D is a bounded cochain complex of projective L -modules. Suppose further that D is R -finitely dominated. Then the induced cochain complex $D \otimes_L R((x_1, x_2, \dots, x_n))$ is acyclic, and thus contractible.*

Proof. By hypothesis there exists a bounded complex C of finitely generated projective R -modules, and mutually inverse R -linear homotopy equivalences $\alpha: C \longrightarrow D$ and $\beta: D \longrightarrow C$. By Corollary III.4.2 we have a homotopy equivalence of L -module complexes

$$\Sigma^n D \simeq \mathcal{T}\mathfrak{D}\mathfrak{ct}(C; \alpha, \beta, G; x_1, x_2, \dots, x_n) =: P ,$$

where $G: \text{id}_D \simeq \alpha \circ \beta$ is a chosen homotopy. Consequently, the complexes remain homotopy equivalent after tensoring (over L) with the NOVIKOV ring $R((x_1, x_2, \dots, x_n))$, and the Theorem will be proved if we can show that the right-hand side becomes acyclic.

To this end, we carry the ideas underpinning §III.3 one step further, employing the notion of truncated product totalisations instead of ordinary totalisation. Let $F = \mathfrak{D}\mathfrak{ct}(C; \alpha, \beta, G; x_1, x_2, \dots, x_n)$ so that $P = \mathcal{T}F$, and let $L(F)^\bullet$ be the multi-complex associated to F as in §III.3. Then $P \cong \text{Tot} L(F)^\bullet$, as observed in Proposition III.3.2. By virtually the same argument, we see that

$$P \otimes_L R((x_1, x_2, \dots, x_n)) \cong {}_{\text{tr}} \text{Tot} L(F)^\bullet ;$$

this uses the cancellation rule

$$M \otimes_R L \otimes_L R((x_1, x_2, \dots, x_n)) \cong M \otimes_R R((x_1, x_2, \dots, x_n))$$

and Lemma III.1.2 (and the fact that every finitely generated projective R -module is finitely presented).

There is a chain of homotopy equivalences of R -module complexes

$$\begin{aligned} 0 &\xrightarrow{\sim} \mathrm{Tot} \mathfrak{I} \mathrm{riv}(D; x_1, x_2, \dots, x_n) \\ &\xleftarrow[\mathrm{II.2.3}]{\sim} \mathrm{Tot} \mathfrak{D} \mathrm{er}(D; \alpha\beta, \mathrm{id}, G; x_1, x_2, \dots, x_n) \\ &\xrightarrow[\mathrm{II.2.6}]{\sim} \mathrm{Tot} \mathfrak{D} \mathrm{er}(C; \alpha, \beta, G; x_1, x_2, \dots, x_n) = \mathrm{Tot}(F) , \end{aligned}$$

the first one from Corollary III.2.2 (as multiplication by x_k is an isomorphism, hence a quasi-isomorphism). This shows that the multi-complex $\mathbb{L}(F)^\bullet$ is acyclic in \mathbf{e}_{n+1} -direction. Consequently its truncated product totalisation is acyclic by Proposition III.2.4. \square

III.6. THE MAIN THEOREM

Cones. Let $M \cong \mathbb{Z}^n$ be a lattice of rank n . The group algebra $R[M]$ is isomorphic to the LAURENT polynomial ring in n indeterminates; a choice of basis b_1, b_2, \dots, b_n of M determines one such isomorphism, with $\pm b_i$ being identified with $x_i^{\pm 1}$. Abstractly, we may think of $R[M]$ as the set of all functions $M \longrightarrow R$ with finite support; the product is defined via the usual convolution type formula.

Let $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ be the n -dimensional \mathbb{R} -vector space associated with M , and note that we have a natural inclusion $M \subset M_{\mathbb{R}}$, $m \mapsto m \otimes 1$. A *rational polyhedral cone* in $M_{\mathbb{R}}$, or shorter just *cone*, is a set $\sigma \subseteq M_{\mathbb{R}}$ for which there exist finitely many elements $v_i \in M$ with

$$\sigma = \mathrm{cone} \{v_1, v_2, \dots, v_\ell\} = \{\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_\ell v_\ell \mid \lambda_i \in \mathbb{R}_{\geq 0}\} .$$

The *dimension* of σ is $\dim \sigma = \dim \mathrm{span}_{\mathbb{R}}(\sigma)$, the \mathbb{R} -dimension of its linear span. The *cospan* of σ is the largest linear subspace contained in σ , which is precisely $\sigma \cap (-\sigma)$. We call σ a *pointed cone* if its cospan is trivial (*i.e.*, consists of the zero vector only). Since our cones are spanned by finitely many vectors, this is equivalent to saying that there exists a hyperplane $H \subset M_{\mathbb{R}}$ with $H \cap \sigma = \{0\}$ such that $\sigma \setminus \{0\}$ is entirely contained in one of the open half spaces determined by H .

Lemma III.6.1. *Given an n -dimensional pointed cone σ there exists a basis of M consisting of elements of $M \cap \sigma$.* \square

Novikov rings determined by cones. An n -dimensional pointed cone σ determines a ring $R((\sigma))$ in the following manner: Choose a vector w in the interior of σ (if σ is spanned by vectors v_i as in the definition of a cone above, then $w = v_1 + v_2 + \dots + v_\ell$ will do), and define

$$R((\sigma)) = \{f : M \longrightarrow R \mid \exists k \geq 0 : kw + \mathrm{supp}(f) \subseteq \sigma\} .^8$$

As before, multiplication is given by a convolution type formula (which makes sense as the cone is pointed). We call $R((\sigma))$ the *NOVIKOV ring associated to σ* . It is independent from the specific choice of vector w .

⁸We write $v + A$ for the set $\{v + a \mid a \in A\}$

Example III.6.2. In $M = \mathbb{Z}^n$ we choose the standard basis e_1, e_2, \dots, e_n consisting of unit vectors. Let σ be the cone spanned by the basis vectors. Then the identification of $R[M]$ with the LAURENT polynomial ring $L = R[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$ extends to an identification of $R((\sigma))$ with the NOVIKOV ring $R((x_1, x_2, \dots, x_n))$ introduced in §III.1.

Suppose now more generally that σ is an n -dimensional cone with possibly non-trivial cospan U . Write $u = \dim(U)$, and define $\bar{M}_{\mathbb{R}} = M_{\mathbb{R}}/U$. The image \bar{M} of M in $\bar{M}_{\mathbb{R}}$ is a lattice of rank $n - u$ such that there is a canonical identification $\bar{M}_{\mathbb{R}} = \bar{M} \otimes_{\mathbb{Z}} \mathbb{R}$, and the image $\bar{\sigma}$ of σ in $\bar{M}_{\mathbb{R}}$ is a pointed $(n - u)$ -dimensional cone. Choose a vector \bar{w} in the interior of $\bar{\sigma}$, and define

$$R((\sigma)) = \{\bar{f}: \bar{M} \longrightarrow R[U \cap M] \mid \exists k \geq 0 : k\bar{w} + \text{supp } \bar{f} \subseteq \bar{\sigma}\} .$$

As before, this is a ring with convolution type product formula; in fact, $R((\sigma)) = R[U \cap M]((\bar{\sigma}))$. We call this the NOVIKOV *ring associated to* σ . It is independent from the specific choice of vector \bar{w} . After choosing a splitting $M = (M \cap U) \oplus \bar{M}$ we can consider $R((\sigma))$ as a subset of the set map (M, R) of maps $M \longrightarrow R$. More explicitly, the ring $R[U \cap M]$ is the set of maps $U \cap M \longrightarrow R$ with finite support; the element $\bar{f}: \bar{M} \longrightarrow R[U \cap M]$ corresponds to the map

$$M = (M \cap U) \oplus \bar{M} \longrightarrow R, \quad m = (u, \bar{m}) \mapsto \bar{f}(\bar{m})(u) .$$

Example III.6.3. In $M = \mathbb{Z}^n$ we choose the standard basis e_1, e_2, \dots, e_n consisting of unit vectors. Let σ be the cone spanned by the basis vectors and the vector $-e_n$. Then the identification of $R[M]$ with the LAURENT polynomial ring $L = R[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$ extends to an identification of $R((\sigma))$ with the NOVIKOV ring $R[x_n^{\pm 1}]((x_1, x_2, \dots, x_{n-1}))$, that is, a NOVIKOV ring in $(n - 1)$ indeterminates over a LAURENT polynomial ring in one indeterminate.

Dual cones. The dual $N = \text{hom}_{\mathbb{Z}}(M, \mathbb{Z})$ of M is again a lattice of rank n ; its associated vector space $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ is naturally identified with the dual $\text{hom}_{\mathbb{R}}(M_{\mathbb{R}}, \mathbb{R})$ of $M_{\mathbb{R}}$. We denote by $\langle \cdot, \cdot \rangle$ the canonical evaluation pairing $M_{\mathbb{R}} \times N_{\mathbb{R}} \longrightarrow \mathbb{R}$. We define (rational polyhedral) cones in $N_{\mathbb{R}}$ just as we did in $M_{\mathbb{R}}$ above. Given a cone σ in $N_{\mathbb{R}}$, we define its *dual* to be the set

$$\sigma^{\vee} = \{x \in M_{\mathbb{R}} \mid \forall y \in \sigma : \langle x, y \rangle \geq 0\} .$$

It can be shown that the dual of a cone is a cone, and that

$$\dim(\sigma^{\vee}) = \text{codim cospan}(\sigma)$$

so that the dual of a pointed cone is an n -dimensional cone. We have $\{0\}^{\vee} = M_{\mathbb{R}}$; the dual of a one-dimensional pointed cone in $N_{\mathbb{R}}$ is a closed half space in $M_{\mathbb{R}}$.

Fans. A fan is a finite complex of pointed cones covering $N_{\mathbb{R}}$. More formally, a (finite, complete) *fan* is a finite collection Δ of pointed cones in $N_{\mathbb{R}}$ such that

- (1) the intersection of any two cones in Δ is a face of either cone, and an element of Δ ;
- (2) $\bigcup_{\sigma \in \Delta} \sigma = N_{\mathbb{R}}$.

The main theorem. We are now in a position to state and prove the main result of this paper: a complete cohomological characterisation of finite domination using the NOVIKOV rings encoded by the non-trivial cones in a given fan.

Theorem III.6.4. *Let D be a bounded cochain complex of finitely generated free $R[M]$ -modules.*

- (a) *Suppose that D is R -finitely dominated.*
 - (i) *The complex D is $R[M \cap U]$ -finitely dominated for any linear subspace $U \subseteq M_{\mathbb{R}}$ which is spanned by elements of M .*
 - (ii) *For every n -dimensional cone $\sigma \neq M_{\mathbb{R}}$ the induced cochain complex $D \otimes_{R[M]} R((\sigma))$ is acyclic.*
- (b) *Let Δ be a (complete) fan. Suppose that for every cone $\{0\} \neq \sigma \in \Delta$ the cochain complex $D \otimes_{R[M]} R((\sigma^\vee))$ is acyclic. Then D is R -finitely dominated.*

Proof. Part (a): We deal with assertion (i) first. Let $U \subseteq M_{\mathbb{R}}$ be a subspace generated by elements of M . As there is nothing to show if $U = \{0\}$ we assume $\dim U \geq 1$. We can choose a basis b_1, b_2, \dots, b_u of U consisting of elements of M . Extend this with $b_{u+1}, b_{u+2}, \dots, b_n$ to a \mathbb{Z} -basis of M . This gives us preferred identifications of rings

$$R[M \cap U] \cong R[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_u^{\pm 1}] =: L'$$

and

$$R[M] \cong R[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_u^{\pm 1}, x_{u+1}^{\pm 1}, \dots, x_n^{\pm 1}] = L.$$

Using this identification we consider D as a complex of L -modules. By restriction of scalars it is also a complex of L' -modules, consisting of free (but not necessarily finitely generated) modules.

By hypothesis on D there are mutually inverse R -linear homotopy equivalences $\alpha: C \rightarrow D$ and $\beta: D \rightarrow C$ with C a bounded cochain complex of finitely generated projective R -modules, and a homotopy $G: \text{id}_D \simeq \alpha \circ \beta$. Then working with the R -algebra L' instead of L we have an L' -linear homotopy equivalence

$$\Sigma^u D \simeq \mathcal{T}\mathcal{D}\mathcal{e}\mathcal{t}(C; \alpha, \beta, G; x_1, x_2, \dots, x_u),$$

by Corollary III.4.2. Now the complex on the right is the totalisation of a homotopy commutative cube on the complex $C \otimes_R L'$, which is bounded and consists of finitely generated projective L' -modules. This shows that $\Sigma^u D$ is L' -finitely dominated, hence so is D .

We now turn our attention to assertion (ii). Let $U = \text{cospan}(\sigma)$, and observe that U is spanned by elements of M (as σ is spanned by elements of M). Write $\bar{\sigma}$ and \bar{M} for the images of σ and M in $\bar{M}_{\mathbb{R}} = M_{\mathbb{R}}/U$ as before. Note that \bar{M} is a lattice of rank $n - u$, and $\bar{\sigma}$ is a cone of dimension $n - u$, where $u = \dim U$ as before.

We have an isomorphism $R((\sigma)) \cong L'((\bar{\sigma}))$. More precisely, a choice of \mathbb{Z} -basis b_1, b_2, \dots, b_n of M as above (so that the $b_k, k \leq u$, form a \mathbb{Z} -basis of $M \cap U$) determines a splitting $M = (M \cap U) \oplus \bar{M}$; with respect to this splitting, $R((\sigma)) = L'((\bar{\sigma}))$ as subsets of $\text{map}(M, R)$.

In fact, we can choose $b_{u+1}, b_{u+2}, \dots, b_n$ in such a way that their respective images in $\bar{M}_{\mathbb{R}}$ lie in $\bar{\sigma}$. For by Lemma III.6.1 we can choose a \mathbb{Z} -basis $\bar{b}_{u+1}, \bar{b}_{u+2}, \dots, \bar{b}_n$ of \bar{M} consisting of elements of $\bar{\sigma} \cap \bar{M}$, and these elements can be lifted along $M \longrightarrow \bar{M}$. With respect to this particular choice of basis, and with respect to the splitting $M = (M \cap U) \oplus \bar{M}$ it entails, we have

$$L'((x_{u+1}, x_{u+2}, \dots, x_n)) \subseteq L'((\bar{\sigma})) = R((\sigma))$$

as subsets of $\text{map}(M, R)$.

Now as the L -module complex D is L' -finitely dominated by (i) we know from Theorem III.5.1, applied to the ground ring L' instead of R , that the complex $D \otimes_L L'((x_{u+1}, x_{u+2}, \dots, x_n))$ is acyclic and, being a bounded complex of free modules, is thus contractible; for this to make sense we also have to note that $L = L'[x_{u+1}, x_{u+2}, \dots, x_n]$, thanks to our choice of basis elements b_k . It follows that tensoring further over $L'((x_{u+1}, x_{u+2}, \dots, x_n))$ with $L'((\bar{\sigma})) = R((\sigma))$ results in a contractible, and thus acyclic, complex. But the result is isomorphic to the complex $D \otimes_L R((\sigma))$, whence assertion (ii) is proved.

Part (b): Let $\xi \in N_{\mathbb{R}}$ be an arbitrary non-zero vector; recall that N is the dual of M so that ξ determines a linear form on $M_{\mathbb{R}}$. The associated set

$$R[M]_{\xi}^{\wedge} = \{f: M \longrightarrow R \mid \forall r \in \mathbb{R} : \#\text{supp}(f) \cap \xi^{-1}(-\infty, r] < \infty\}$$

carries a ring structure with multiplication given by convolution. By a theorem of SCHÜTZ [Sch06, Theorem 4.7] it will be enough to show that $D \otimes_{R[M]} R[M]_{\xi}^{\wedge}$ is acyclic. But this is easy: as our fan covers all of $N_{\mathbb{R}}$ there is a (unique) smallest cone $\sigma \in \Delta$ with $\xi \in \sigma$. “Smallest” here means with respect to inclusion, or equivalently with respect to dimension; in any case, σ is the (unique) cone that contains ξ in its (relative) interior (and not in one of its proper faces). As $\xi \neq 0$ we have $\sigma \neq \{0\}$. Then $R((\sigma)) \subseteq R[M]_{\xi}^{\wedge}$ and

$$D \otimes_{R[M]} R[M]_{\xi}^{\wedge} \cong D \otimes_{R[M]} R((\sigma)) \otimes_{R((\sigma))} R[M]_{\xi}^{\wedge}.$$

But $D \otimes_{R[M]} R((\sigma))$ is acyclic by hypothesis, hence contractible (since D is bounded and consists of free modules); consequently, $D \otimes_{R[M]} R[M]_{\xi}^{\wedge}$ is contractible and thus acyclic as well. \square

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